Not Formulae and Statistical Tables for GCE Mathematics and GCE Statistics

The Not-Formula Book

Core 2

Everything you need to remember that the formula book won't tell you

The Not-Formula Book for C2

Everything you need to know for Core 2 that *won't* be in the formula book

Examination Board: AQA

Brief

This document is intended as an aid for revision. Although it includes some examples and explanation, it is primarily not for learning content, but for becoming familiar with the requirements of the course as regards formulae and results. It cannot replace the use of a text book, and nothing produces competence and familiarity with mathematical techniques like practice. This document was produced as an addition to classroom teaching and textbook questions, to provide a summary of key points and, in particular, any formulae or results you are expected to know and use in this module.

Contents

Chapter 1 – Indices

The **multiplication** rule:

$$a^m \times a^n = a^{m+n}$$

The **division** rule:

$$a^m \div a^n = a^{m-n}$$

The **power** rule:

$$(a^m)^n = a^{mn}$$

The **negative index** rule:

$$a^{-n} = \frac{1}{a^n}$$

The **zero index** result:

$$a^0 = 1$$

The **root** rule:

$$a^{\frac{1}{n}} = \sqrt[n]{a}$$

In general:

$$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$$

Eg:

Simplify: $\frac{\sqrt[4]{x^2y^3}}{x^{a-3}}$

$$\frac{\sqrt[4]{x^2y^3}}{x^3} = x^{\frac{2}{4}}y^{\frac{3}{4}}x^{-3} = x^{\frac{1}{2}-3}y^{\frac{3}{4}} = \boldsymbol{x^{-\frac{5}{2}}y^{\frac{3}{4}}}$$

Eg 2:

Solve: $\frac{3^{3x}}{27} = \frac{1}{9^{2x+1}}$

$$\frac{3^{3x}}{27} = \frac{1}{9^{2x+1}} \quad \Longrightarrow \quad \frac{3^{3x}}{3^3} = 9^{-2x-1}$$

$$\Longrightarrow \quad 3^{3x-3} = (3^2)^{-2x-1} \quad \Longrightarrow \quad 3^{3x-3} = 3^{-4x-2}$$

$$\Longrightarrow \quad 3x - 3 = -4x - 2 \quad \Longrightarrow \quad 7x = 1 \quad \Longrightarrow \quad x = \frac{1}{7}$$

Chapter 2 – Further differentiation

The differentiation methods introduced in C1 are all applicable here. Terms can be simplified using the index rules from chapter 1 to rearrange expressions in order to differentiate them.

Eg: Find the gradient of the curve $y = \frac{3}{x} - \frac{4}{x^2} + 36\sqrt{x}$ at the point where $x = 4$.

$$y = \frac{3}{x} - \frac{4}{x^2} + 36\sqrt{x} = 3x^{-1} - 4x^{-2} + 36x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -3x^{-2} + 8x^{-3} + 18x^{-\frac{1}{2}} = -\frac{3}{x^2} + \frac{8}{x^3} + \frac{18}{\sqrt{x}}$$

$$x = 4 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{3}{16} + \frac{8}{64} + \frac{18}{2} = \frac{572}{64} = 8.9375$$

Recall that a **stationary point** is a point on the curve where the **gradient** is 0.

$$\frac{dy}{dx} = 0 \quad and \quad \frac{d^2y}{dx^2} > 0 \quad \Longrightarrow \quad Minimum \qquad \frac{dy}{dx} = 0 \quad and \quad \frac{d^2y}{dx^2} < 0 \quad \Longrightarrow \quad Maximum$$

You have already come across **optimisation** in C1, but should be able to deal with problems involving negative or fractional indices now.

Eg: A square-based cuboidal display box has an open top and open front. It has a volume of $18\ litres$, or $18000cm^3$. Find the minimum surface area and the corresponding optimal height.

Let the width of the base be x, and the height h. First we eliminate h by using the fixed volume:

$$x^2h = 18000 \quad \Longrightarrow \quad h = \frac{18000}{x^2}$$

Next, calculate the surface area of the cardboard:

$$Base: \quad x^2 \quad Back + Left + Right: \quad 3xh = \frac{54000}{x} \quad \Longrightarrow \quad S = x^2 + \frac{54000}{x} = x^2 + 54000x^{-1}$$

To minimise this, calculate $\frac{dS}{dx}$ and use it to find any stationary points:

$$\frac{dS}{dx} = 0 \quad \Longrightarrow \quad \frac{dS}{dx} = 2x - 54000x^{-2} = 2x - \frac{54000}{x^2} = 0$$

$$\Longrightarrow \quad 2x^3 - 54000 = 0 \quad \Longrightarrow \quad x = \sqrt[3]{27000} = 30$$

Substitute this into our expression for S, then use it to find h:

$$Minimum\ S = 30^2 + \frac{54000}{30} = 10800cm^2 \qquad Optimal\ h = \frac{18000}{30^2} = 20cm$$

Chapter 3 – Further integration and the trapezium rule

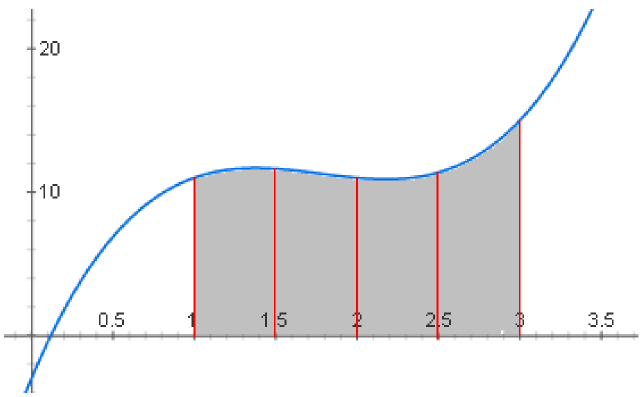
As with differentiation, we can now integrate expressions other than simply polynomials.

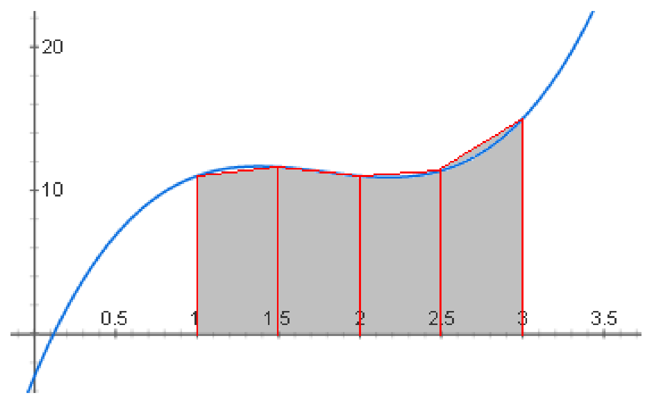
Eg: Find the area under the curve $y = \sqrt[3]{x}\left(2x^3 - \frac{1}{x}\right)$ between the points $x = 1$ and $x = 2$.

$$\int_1^2 \sqrt[3]{x}\left(2x^3 - \frac{1}{x}\right) dx = \int_1^2 x^{\frac{1}{3}}(2x^3 - x^{-1})\, dx = \int_1^2 2x^{\frac{10}{3}} - x^{-\frac{2}{3}}\, dx$$

$$= \left[\frac{6}{13}x^{\frac{13}{3}} - 3x^{\frac{1}{3}}\right]_1^2 = \left(\frac{6}{13}(2)^{\frac{13}{3}} - 3(2)^{\frac{1}{3}}\right) - \left(\frac{6}{13}(1)^{\frac{13}{3}} - 3(1)^{\frac{1}{3}}\right) = 8.06\ to\ 3\ s.f.$$

If a function is complicated, it can be difficult to integrate to find the exact value of the area beneath the curve. In this case we can use the **trapezium rule** to find an **estimate** for the area.





Note: The trapezium rule is based on the idea that as ordinates get closer together, the straight line joining them becomes close to the curve. Integration in general is simply an extension of this idea.

$$\int_a^b f(x)\, dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})\} \quad where\ h = \frac{b-a}{n}$$

Note: This formula **is** in the formula book, but is included here for completeness. You need to be confident interpreting and applying it.

Eg: Use the trapezium rule with five ordinates (four strips) to estimate the area under the curve $y = 10^x$ between $x = 2$ and $x = 4$.

$$h = \frac{4-2}{4} = \frac{1}{2}$$

$x_0 = 2$ $\quad y_0 = 10^2 = 100$
$x_1 = 2.5$ $\quad y_1 = 10^{2.5} = 316.22 \ldots$
$x_2 = 3$ $\quad y_2 = 10^3 = 1000$
$x_3 = 3.5$ $\quad y_3 = 10^{3.5} = 3162.27 \ldots$
$x_4 = 4$ $\quad y_4 = 10^4 = 10000$

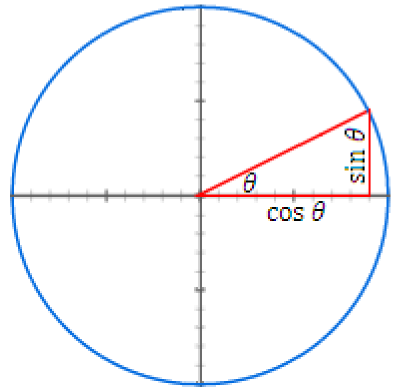
$$\int_2^4 10^x\, dx \approx \frac{1}{2}\left(\frac{1}{2}\right)\{(100 + 10000) + 2(316.22 \ldots + 1000 + 3162.27 \ldots)\} = 4764.25 \ldots$$

Chapter 4 – Basic trigonometry

The *sine* function can be thought of as the height above the centre of a point moving around a circle. The *cosine* function can be thought of as the distance to the right of the centre.

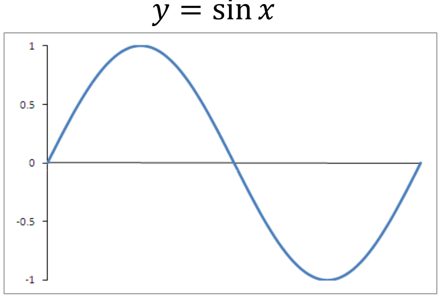
The angle in question is the anti-clockwise angle from the x axis as shown.

Note that the height (the *sine* function) will be negative between 180° and 360°, and the distance to the right (the *cosine* function) will be negative between 90° and 270°.

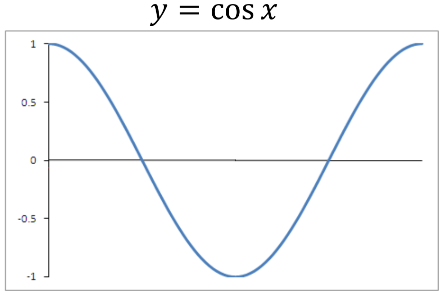


It is important to note that the functions $\sin x$, $\cos x$ and $\tan x$ are valid not just for angles from 0° to 90°. Using the idea of the circle above, we can generate graphs for each function. Also, since it is possible to go as far clockwise or anticlockwise around the circle, the function, and therefore graph, extends infinitely in both directions, but will simply repeat the first 360°.

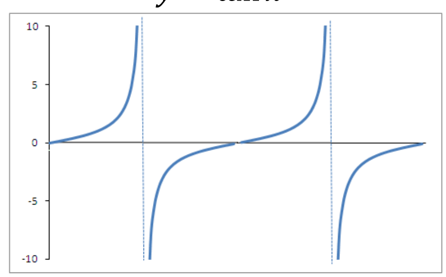
$y = \sin x$



$y = \cos x$



$y = \tan x$



From the **graph** of $\sin x$, we can deduce that:

$$\sin(360° + \theta) = \sin\theta \quad \sin(360° - \theta) = -\sin\theta$$

$$\sin(180° - \theta) = \sin\theta \quad \sin(180° + \theta) = -\sin\theta$$

$$\sin(-\theta) = -\sin\theta$$

The **solutions** to the **equation** $\sin\theta = \sin\alpha$ for the range $0° \le \alpha \le 360°$ are:

$$\theta = \alpha \quad and \quad \theta = 180° - \alpha$$

Note: These two solutions are merely those within the range $0° \le \alpha \le 360°$. Using the results above, an infinite set of solutions can be found by simply adding or subtracting multiples of 360°.

From the **graph** of $\cos x$, we can deduce that:

$$\cos(360° + \theta) = \cos\theta \quad \cos(360° - \theta) = \cos\theta$$

$$\cos(180° + \theta) = -\cos\theta \quad \cos(180° - \theta) = -\cos\theta$$

$$\cos(-\theta) = \cos\theta$$

The **solutions** to the **equation** $\cos\theta = \cos\alpha$ for the range $-180° \leq \alpha \leq 180°$ are:

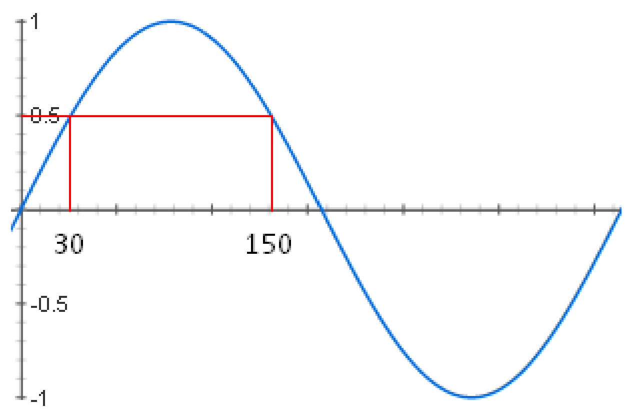
$$\theta = \alpha \quad and \quad \theta = -\alpha$$

Note: The key thing to remember is the general shape of both $y = \sin x$ and $y = \cos x$. Note that the *cosine* graph is symmetrical about the y axis.

Eg: Find all solutions to the equation $\sin\theta = \sin 30$ in the range $0° \leq \theta \leq 360°$.

$$\sin\theta = \sin(180 - \theta)$$

$$\sin 30 = \sin 150$$



From the **graph** of $\tan x$, we can deduce that:

$$\tan(360° + \theta) = \tan\theta \quad \tan(360° - \theta) = -\tan\theta$$

$$\tan(180° + \theta) = \tan\theta \quad \tan(180° - \theta) = -\tan\theta$$

$$\tan(-\theta) = -\tan\theta$$

The **solutions** to the **equation** $\tan\theta = \tan\alpha$ for the range $0° \leq \alpha \leq 360°$ are:

$$\theta = \alpha \quad and \quad \theta = \alpha + 180°$$

Note: The graph of $y = \tan x$ has asymptotes at 90°, 270° and multiples of 360° either side.

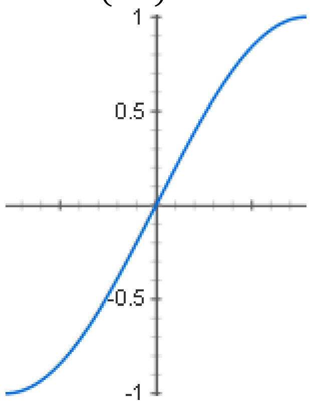
Due to the cyclical nature of trig functions, while a calculator can give you the value of $\cos\theta$ for any angle θ, **working backwards** gives an infinite number of possible angles. The calculator will provide a **principal value** for an inverse trig function, which will typically be the closest to 0°.

Principle values for $\sin\theta$ will lie between $-90°$ and $90°$.

Eg:

$$\sin^{-1}(0.5) = 30°$$

$$\sin^{-1}(-1) = -90°$$

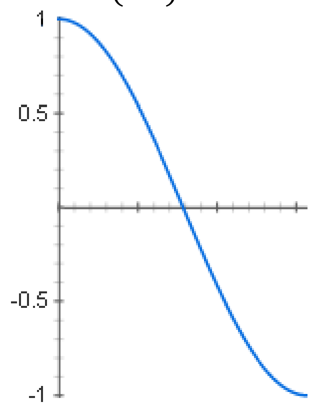


Principle values for $\cos\theta$ will lie between $0°$ and $180°$.

Eg:

$$\cos^{-1}(0.5) = 60°$$

$$\cos^{-1}(-1) = 180°$$

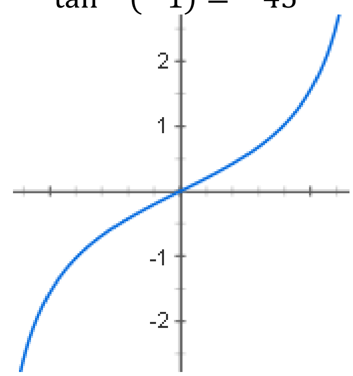


Principle values for $\tan\theta$ will lie between $-90°$ and $90°$.

Eg:

$$\tan^{-1}(0.5) = 26.565\ldots°$$

$$\tan^{-1}(-1) = -45°$$



Note: More valid solutions can be found by considering the graph for the range in question.

Eg: Find all solutions for $\sin\theta = -0.3$ in the range $0 \leq \theta \leq 360$.

Find principle value using a calculator:

$$\sin^{-1}(0.3) = -17.46° \; to \; 2 \; d.p.$$

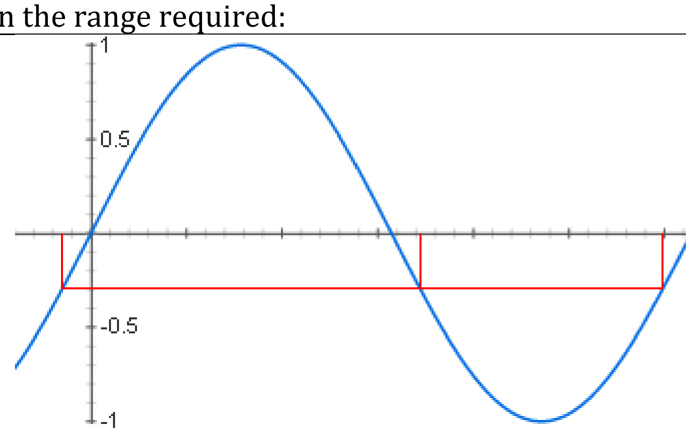
Consider the graph to identify all solutions within the range required:

To find the right-most solution in the range, which is one complete cycle to the right, add 360° to the principle value:

$$-17.46 + 360 = 342.54° \; to \; 2 \; d.p.$$

To find the left-most solution in the range, which is the same distance to the right of 180° as our principle value is to the left of 0, we effectively subtract our (negative) principle value from 180°:

$$180 - (-17.46) = 197.46° \; to \; 2 \; d.p.$$



Note: These solutions can be found using the results previously quoted: $\sin(360° + \theta) = \sin\theta$ and $\sin(180 - \theta) = \sin\theta$, but it is often easiest to visualise the values on the graph instead.

Solving equations of the form $\sin x = k$, $\cos x = k$ or $\tan x = k$ will generally yield **two solutions** within any given 360° interval, (provided $-1 \leq k \leq 1$ for sin and cos. k can be anything for tan).

Note: The only exceptions to this rule are end-points of the range. For instance, for the range $0 \leq x \leq 360$, the equation $\sin x = 0$ has three solutions: $x = 0°$, $x = 180°$ and $x = 360°$. Be careful to note the nature of the inequality signs in such cases ($\sin x = 0$ could have 1, 2 or 3 solutions depending on whether the signs are $\leq$ or $<$).

The **sine rule** which will have already been introduced at GCSE is now extended to:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

For a triangle with angles A, B and C, opposite sides of length a, b and c respectively, and whose circumcircle (the unique circle which can be drawn through all 3 corners) has radius R.

Eg: A triangle is constructed within a circle of radius $10cm$, and has two side lengths $8cm$ and $12cm$. Given that the angle between the two known sides is obtuse, find all three angles and the length of the remaining side.

Using $a = 8$, $b = 12$ and $R = 10$, and substituting into the formula:

$$\frac{8}{\sin A} = \frac{12}{\sin B} = \frac{c}{\sin C} = 20$$

Choosing an appropriate equality to examine:

$$\frac{8}{\sin A} = 20 \quad \Longrightarrow \quad \sin A = \frac{8}{20} = 0.4 \quad \Longrightarrow \quad A = 23.6° \text{ to } 1\ d.p.$$

Note: We know $\theta < 90°$ from the question, therefore we can use the principle value for the angle. Otherwise, a second solution of $180 - 23.6 = 156.4°$ would be a possibility.

Finding angle B in the same way:

$$\frac{12}{\sin B} = 20 \quad \Longrightarrow \quad \sin B = \frac{12}{20} = 0.6 \quad \Longrightarrow \quad B = 36.9° \text{ to } 1\ d.p.$$

Finding angle C:

$$180 - (36.9 + 23.6) = 119.6° \text{ to } 1\ d.p.$$

Note: While angles are quoted to $1\ d.p.$, the exact values have been stored on a calculator and are used in subsequent calculations.

Finding the remaining side:

$$\frac{c}{\sin 119.6} = 20 \quad \Longrightarrow \quad c = 20 \sin 119.6 = 17.4cm \text{ to } 1\ d.p.$$

Note: In the previous example we were given the information that a particular angle was obtuse. Without this additional condition there could be two possible triangles which fit the requirements. This is called the ambiguous case, and must sometimes be dealt with by finding both cases which fit the criteria.

Eg: A triangle has two sides $3cm$ and $8cm$, and an angle of $20°$ opposite the $3cm$ side. Find the two possible lengths of the third side.

Use formula to find two possible values for the angle opposite the $8cm$ side:

$$\frac{3}{\sin 20} = \frac{8}{\sin B} \quad \Longrightarrow \quad \sin B = \frac{8 \sin 20}{3} = 0.912 \ldots \quad \Longrightarrow \quad B = 65.8° \quad or \quad 114.2° \; to\ 1\ d.p.$$

Find the two possibilities for the third angle of the triangle:

$$C = 180 - (20 + 65.8) = 94.2° \quad or \quad 180 - (45 + 114.2) = 20.8°$$

Use these two possibilities to find two values for the third side:

$$\frac{c}{\sin C} = \frac{3}{\sin 20} \quad \Longrightarrow \quad c = \frac{3 \sin C}{\sin 20} = \frac{3 \sin 94.2}{\sin 20} = 8.75cm \quad or \quad c = \frac{3 \sin 20.8}{\sin 20} = 3.11cm$$

The **cosine rule** links the lengths of the three sides of a triangle to an angle in the triangle:

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$where\ a, b\ and\ c\ are\ side\ lengths, and\ A\ is\ the\ angle\ opposite\ side\ a$$

Eg: Find the smallest angle of a triangle with side lengths $7cm$, $8cm$ and $9cm$.

The smallest angle will be opposite the smallest side, so using $a = 7$, $b = 8$, $c = 9$ and finding angle A:

$$7^2 = 8^2 + 9^2 - 2(8)(9) \cos A \quad \Longrightarrow \quad \cos A = \frac{64 + 81 - 49}{144} \quad \Longrightarrow \quad A = 48.2°$$

The **area of a triangle** can be calculated using:

$$A = \frac{1}{2} ab \sin C$$

$$where\ a\ and\ b\ are\ the\ lengths\ of\ two\ sides\ of\ a\ triangle\ and\ C\ is\ the\ angle\ in\ between$$

Eg: The right-angled triangle with side lengths $6cm$, $8cm$ and $10cm$ has the same area as a triangle with two of its sides of length $5cm$ and $12cm$. Calculate the angle between these two sides.

Area of right-angled triangle:

$$A = \frac{bh}{2} = \frac{6 \times 8}{2} = 24cm^2$$

Area of second triangle:

$$A = \frac{1}{2} ab \sin C = \left(\frac{1}{2} \times 5 \times 12\right) \sin C = 30 \sin C$$

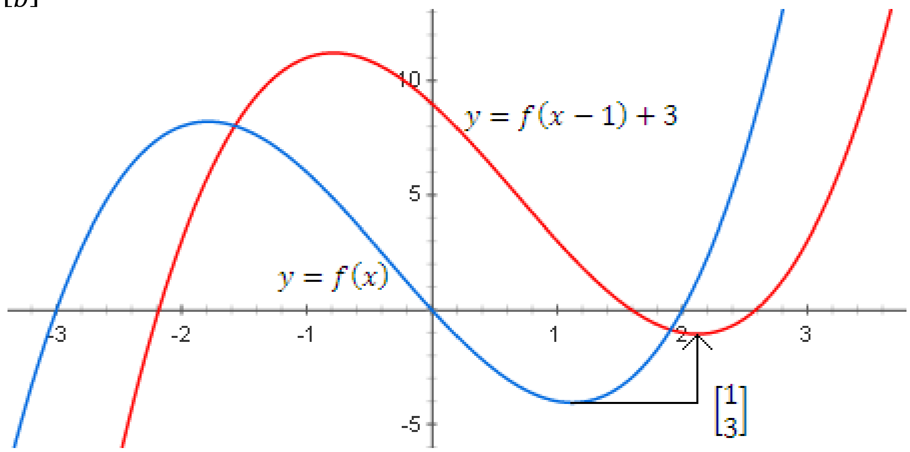
Equating the two:

$$30 \sin C = 24 \quad \Longrightarrow \quad \sin C = \frac{24}{30} = 0.8 \quad \Longrightarrow \quad C = 53.1°$$

Chapter 5 – Simple transformation of graphs

By modifying the input (x) values and output (y) values of a function, we can transform the shape and position of graphs in a predictable fashion.

A **translation** of $\begin{bmatrix} a \\ b \end{bmatrix}$ transforms the graph $y = f(x)$ into the graph $y = f(x - a) + b$.



Note: While adding a number to the end of a function seems a fairly intuitive way of moving it up, *subtracting* a number from the x part to move to the *right* seems somewhat counter-intuitive. Make sure you are confident with which way round this works, and are happy changing a function in this way.

Eg: Write down the equation of the curve produced when $y = 3x^2 - x + 5$ is translated by $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

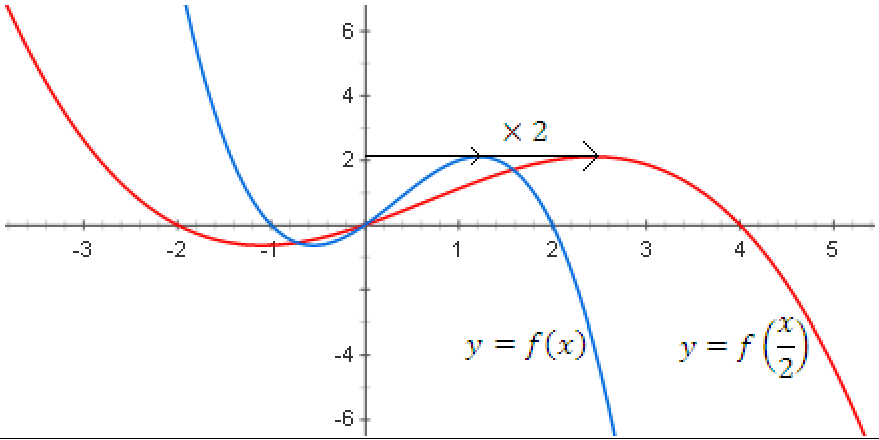
Since we are moving *left*, we will be *adding* to the x values. Replace each x with $x - (-2)$ or $x + 2$:

$$y = 3(x + 2)^2 - (x + 2) + 5 = 3(x + 2)^2 - x + 3 = 3x^2 + 12x + 4 - x + 3 = 3x^2 + 11x + 7$$

Finally, add 4 to the function to move up:

$$y = 3x^2 + 11x + 11$$

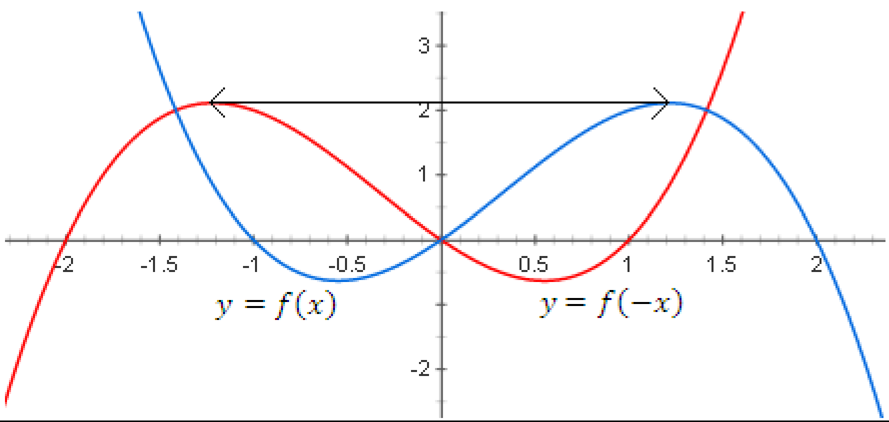
A **stretch** of scale factor a in the x-direction (horizontally) transforms the graph $y = f(x)$ into the graph $y = f\left(\frac{x}{a}\right)$.



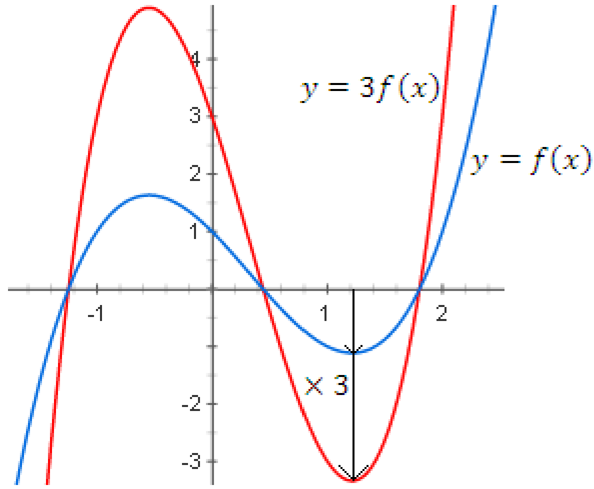
Note: A stretch in the x direction increases the distance from the y axis (the line $x = 0$) by the scale factor. Also note that the rule for a *stretch* in the x direction is also counter-intuitive, unlike the rule for a stretch in the y direction. Replacing x with $4x$, for instance, represents a scale factor of $\frac{1}{4}$, not 4.

A **reflection** in the y axis is equivalent to a **stretch of scale factor – 1** in the x direction:

$$y = f(x) \quad is\ transformed\ into \quad y = f(-x)$$



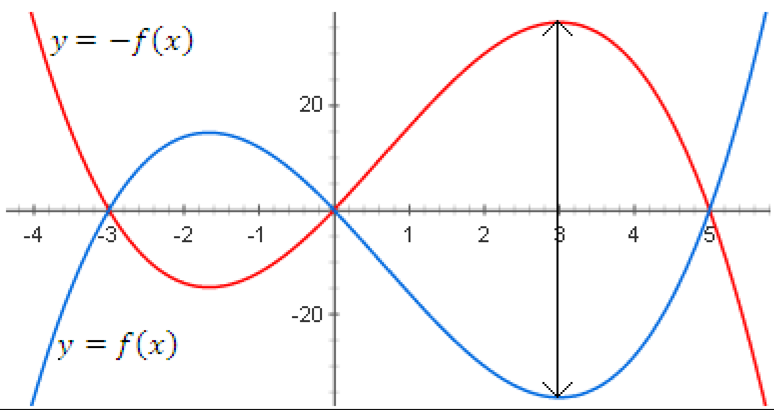
A **stretch** of scale factor a in the y-direction (vertically) transforms the graph $y = f(x)$ into the graph $y = af(x)$.



Note: A stretch in the y direction increases the distance from the x axis ($y = 0$) by the scale factor.

A **reflection** in the x axis is equivalent to a **stretch of scale factor – 1** in the y direction:

$$y = f(x) \quad is\ transformed\ into \quad y = -f(x)$$



Note: Depending on the specific transformations being applied, the order may be important. When asked to describe the transformations, it may be necessary to rearrange in order to determine the type and details of the transformation. There may also be more than one equivalent transformation for a particular function.

Eg: The function $y = 5^x$ is transformed into the function $y = 25(5^x)$. Describe two different transformations that this could represent.

$$Transformation\ of\ the\ form\ y = af(x) \quad \Longrightarrow \quad Stretch\ of\ scale\ factor\ 25\ in\ the\ y\ direction$$

Or:

$$y = 25(5^x) = 5^2 \times 5^x = 5^{x+2} \quad is\ of\ the\ form\ y = f(x-a) \quad \Longrightarrow \quad Translation\ by\ \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Chapter 6 – Solving trigonometrical equations

To **solve an equation** of the form $\mathbf{sin}(\boldsymbol{bx}) = \boldsymbol{k}$, etc, first change the limits of the solution interval so they relate to bx, find all valid solutions for bx, then convert back to x.

Eg: Find all the solutions to $\cos 3x = 0.5$ in the interval $0° \leq x \leq 180°$.

Change the interval to relate to $3x$:

$$0° \leq x \leq 180° \quad \Longrightarrow \quad 0° \leq 3x \leq 540°$$

Find a primary solution for $3x$:

$$3x = \cos^{-1} 0.5 = 60°$$

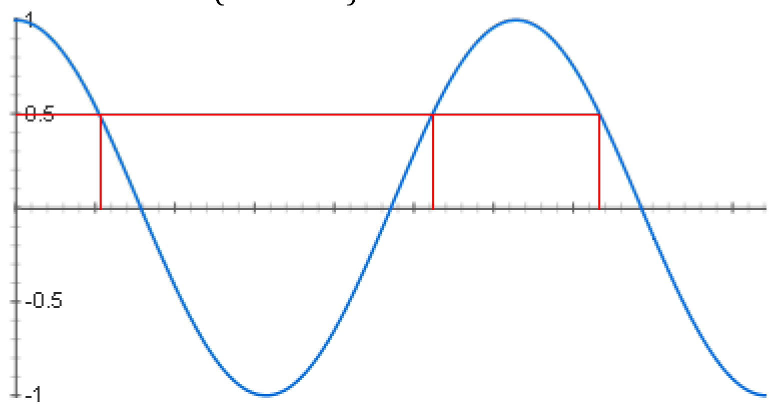
Use the cosine graph to find any other solutions within the (modified) interval:

$$3x = 60°$$

$$3x = 60 + 360 = 420°$$

$$3x = 360 - 60 = 300°$$

Note: You can use your calculator to verify $\cos 420 = 0.5$ and $\cos 300 = 0.5$



Finally, divide by 3 to give all solutions for x within the original interval:

$$x = \frac{60}{3} = 20° \quad and \quad x = \frac{420}{3} = 140° \quad and \quad x = \frac{300}{3} = 100°$$

Note: You should make sure at the end that each of these solutions lies within the original interval, and that each of them solves the original equation.

To **solve an equation** of the form $\mathbf{sin}(\boldsymbol{x} + \boldsymbol{b}) = \boldsymbol{k}$, etc, first change the limits of the solution interval so they relate to $x + b$, find all valid solutions for $x + b$, then convert back to x.

Eg: Find all solutions to $\tan(x + 100) = 2$ in the interval $0° \leq x \leq 360°$.

$$-90° \leq x \leq 90° \quad \Longrightarrow \quad 100° \leq x + 100 \leq 460°$$

$$Primary\ Solution: \ \ x + 100 = \tan^{-1} 2 = 63.4°\ to\ 1\ d.p.$$

$$Valid\ solutions\ from\ graph: \ \ \ x + 100 = 243.4° \quad and \quad 423.4°$$

$$Convert\ for\ x: \ \ x = 143.4° \quad and \quad x = 323.4°$$

By considering the **circle approach** to trigonometry, as introduced in chapter 4, and applying Pythagoras' theorem, it can be shown that:

$$\sin^2\theta + \cos^2\theta = 1$$

Note: It is often useful to write this formula as $\sin^2\theta = 1 - \cos^2\theta$ or $\cos^2\theta = 1 - \sin^2\theta$, and as a result equations involving the *sine* function can be converted, where needed, into equations involving the *cosine* function.

It is also possible to demonstrate that:

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Note: Any problems involving trigonometrical equations will rely on knowledge of these two identities, so it is crucial that you are not only familiar with them, but recognise the situations when one or other may be beneficial.

Eg: Find all solutions to the equation $2\sin x = \cos x$ in the interval $0° \leq x \leq 360°$.

$$2\sin x = \cos x \quad \Longrightarrow \quad \frac{\sin x}{\cos x} = \frac{1}{2} \quad \Longrightarrow \quad \tan x = \frac{1}{2}$$

$$\textit{Primary solution}: \quad x = 26.6° \quad \textit{All solutions (from graph)}: \quad x = 26.6° \quad \textit{and} \quad x = 206.6°$$

Eg: Show that $\sin^2 x - \cos^2 x = 1 - 2\cos^2 x$.

$$\sin^2 x - \cos^2 x = (1 - \cos^2 x) - \cos^2 x = 1 - 2\cos^2 x$$

Eg: Find all solutions to the equation $2\sin^2 x - \cos x - 1 = 0$ in the interval $0° \leq x \leq 360°$.

$$2\sin^2 x - \cos x - 1 = 0 \quad \Longrightarrow \quad 2(1 - \cos^2 x) - \cos x - 1 = 0$$

$$\Longrightarrow \quad 2 - 2\cos^2 x - \cos x - 1 = 0 \quad \Longrightarrow \quad 2\cos^2 x + \cos x - 1 = 0$$

$$\Longrightarrow \quad (2\cos x - 1)(\cos x + 1) = 0 \quad \Longrightarrow \quad \cos x = \frac{1}{2} \quad or \quad \cos x = -1$$

Solutions from $\cos x = \frac{1}{2}$:

$$x = 60°, 300°$$

Solutions from $\cos x = -1$:

$$x = 180°$$

All solutions:

$$x = 60°, 180°, 300°$$

Chapter 7 – Factorials and binomial expansions

The **factorial** function calculates the number of permutations possible for a given number of objects:

$$n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1$$

Eg: How many different orders could a hand of 5 different cards be arranged into?

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

If we know the value of $n!$, we can calculate $(n+1)!$ **inductively** using:

$$(n+1)! = (n+1)n!$$

Eg: $9! = 362880$. Find $10!$.

$$10! = 10 \times 9! = 10 \times 362880 = 3628800$$

Using the above definition we can **work backwards** to define 0! as:

$$1! = 1 \times 0! \quad \Rightarrow \quad 0! = 1$$

To calculate the number of ways of **choosing** r different objects from a total of n objects, use:

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

Eg: A team of 4 students must be chosen from a class of 7. How many different teams can there be?

$$\binom{7}{4} = \frac{7!}{4!\,(7-4)!} = \frac{7!}{4!\,3!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(4 \times 3 \times 2 \times 1)(3 \times 2 \times 1)} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$$

In general, the combinations formula is **symmetric**:

$$\binom{n}{r} = \binom{n}{n-r}$$

Note: This makes more sense if you consider the example above. If I were choosing a team of 4 from a class of 7, for every possible team there is a group of 3 students who are not chosen. Therefore there must be the same number of ways of choosing teams of 3 from 7 as there are choosing 4 from 7.

Pascal's triangle is a quick way of finding values of $\binom{n}{r}$ for small values of n and r:

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1
\end{array}
$$

As an example, the latest row is equivalent to:

$$\binom{5}{0} \ \binom{5}{1} \ \binom{5}{2} \ \binom{5}{3} \ \binom{5}{4} \ \binom{5}{5}$$

Note: Each value can be found by adding the two values above. There are a lot of patterns that emerge by looking at diagonal lines of numbers (eg triangular numbers).

When **multiplying out brackets** with large powers, Pascal's triangle can be used to identify coefficients of successive terms.

Eg: Multiply out $(2+x)^5$.

$$(2+x)^5 = (1)2^5 + (5)2^4x + (10)2^3x^2 + (10)2^2x^3 + (5)2x^4 + (1)x^5$$

$$= 32 + 80x + 80x^2 + 40x^3 + 10x^4 + x^5$$

In general:

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{r}a^{n-r}b^r + \cdots + b^n$$

where n is a positive integer

Note: This formula **is** in the formula book. It is included here for completeness.

Eg: Find the coefficient of x^6 in the expansion of $(2x+3)(x+4)^7$.

Find the x^5 and x^6 term for the bracket with the power:

$$x^5: \quad (x+4)^7 = \cdots + \binom{7}{5}4^2x^5 + \binom{7}{6}4x^6 + \cdots = \cdots + 336x^5 + 28x^6 + \cdots$$

Multiply by the x term and the constant, respectively, to give the resulting x^6 term:

$$336x^5 \times 2x = 672x^6 \qquad and \qquad 28x^6 \times 3 = 84x^6$$

Add to get the resulting x^6 coefficient:

$$672x^6 + 84x^6 = 756x^6 \quad \Rightarrow \quad Coefficient\ of\ x^6\ is\ 756$$

Chapter 8 – Sequences and series

A **sequence** is a list of numbers, eg:

$$4 \quad 5.5 \quad 7 \quad 8.5 \quad 10 \quad \ldots$$

A **series** is the sum of such a list, eg:

$$3 + 6 + 12 + 24 + \cdots$$

The **first term** of a sequence is often written as t_1 or U_1, and the n^{th} term as t_n or U_n.

A sequence can be defined **inductively** by constructing a rule linking each term to a previous term.

Eg: Find the first 4 terms of the sequence defined by $U_n = 2U_{n-1} + 1, U_1 = 4$.

$$U_1 = 4 \quad U_2 = 2(4) + 1 = 9 \quad U_3 = 2(9) + 1 = 19 \quad U_4 = 2(19) + 1 = 39$$

If a sequence can be defined inductively as $U_n = U_{n-1} + d$ for some common difference d, it is known as an **arithmetic sequence**.

Eg: The sequence 6 11 16 21 26 … is an arithmetic sequence.

The $\boldsymbol{n^{th}}$ **term** of an **arithmetic sequence** is given by:

$$U_n = a + (n-1)d$$

where a is the first term and d is the common difference.

One of the simplest **arithmetic series** (the *sum* of a sequence) is the series of **natural numbers**.

The sum of the first n natural numbers is given by:

$$S_n = \frac{n}{2}(n-1)$$

Note: This rule will calculate the sum of a series such as $1 + 2 + 3 + 4 + 5$, but can be extended.

Eg: Find the sum of the even numbers between 900 and 1000.

The even numbers between 900 and 1000: $900 + 902 + 904 + \cdots + 1000$

Firstly, take out 2 as a factor: $2(450 + 451 + 452 + \cdots + 500)$

Secondly, notice that this is the same as: $2\{(1 + 2 + 3 + \cdots + 500) - (1 + 2 + 3 + \cdots + 449)\}$

Finally, use the formula: $2\left\{\frac{500}{2}(501) - \frac{449}{2}(450)\right\} = 48450$

The sum of the first n terms of an **arithmetic series** is given by:

$$S_n = \frac{n}{2}(a + l) \quad or \quad S_n = \frac{n}{2}(2a + (n - 1)d)$$

where a is the first term, l is the last term and d is the common difference.

Note: These two formulae **are** in the formula book. They are included here for completeness.

Note: You will need to be able to effectively apply the formulae for U_n and S_n. This may involve solving simultaneous equations to find a and d.

Eg: The third and seventh terms of an arithmetic series are 12 and 52 respectively. Calculate the sum of the first five terms.

Construct equations from the information provided:

$$U_3 = a + 2d \quad \Longrightarrow \quad 12 = a + 2d \quad and \quad U_7 = a + 6d \quad \Longrightarrow \quad 52 = a + 6d$$

Solve simultaneously (this is easy to do in these situations using elimination):

$$4d = 40 \quad \Longrightarrow \quad d = 10 \quad \Longrightarrow \quad a = -8$$

Use the formula to find S_5:

$$S_5 = 2(-8) + 4(10) = 24$$

If the r^{th} term of a series is U_r then the sum of the first n terms can be written using **sigma notation**:

$$S_n = \sum_{r=1}^{n} U_r$$

Eg: Evaluate $\sum_5^8 (n^2 - n)$.

$$\sum_5^8 (n^2 - n) = (5^2 - 5) + (6^2 - 6) + (7^2 - 7) + (8^2 - 8) = 148$$

Note: The example above could be written as $(5^2 + 6^2 + 7^2 + 8^2) - (5 + 6 + 7 + 8)$.

In general:

$$\sum aF(r) + bG(r) + c = a\sum_{r=1}^{n} F(r) + b\sum_{r=1}^{n} G(r) + cn$$

Note: If a constant is involved, such as c above, since you are summing n terms, each of which has $+c$ at the end, the overall effect is simply adding n lots of c, or cn.

Using sigma notation, the sum of the first n natural numbers result can be written as:

$$\sum_{r=1}^{n} r = \frac{n}{2}(n+1)$$

Note: This formula **is** in the formula book. It is included here for completeness – you must be confident applying it.

Note: Although there is a $\frac{1}{2}$ in this formula, it will always give a whole number answer, as either n or $n+1$ must be even.

Note: You could be required to interpret problems involving arithmetic series written using sigma notation.

Eg: Evaluate $\sum_{r=10}^{20}(5r-4)$.

Method 1 – Splitting up and using $\frac{n}{2}(n+1)$:

$$\sum_{r=10}^{20}(5r-4) = \sum_{r=1}^{20}(5r-4) - \sum_{r=1}^{9}(5r-4) = \left(5\sum_{r=1}^{20} r - 4(20)\right) - \left(5\sum_{r=1}^{9} r - 4(9)\right)$$

$$= \left(5\left(\frac{20}{2}(21)\right) - 80\right) - \left(5\left(\frac{9}{2}(10)\right) - 36\right) = 970 - 189 = 781$$

Method 2 – Using arithmetic series:

$$\sum_{r=10}^{20}(5r-4) = 46 + 51 + \cdots \quad \Rightarrow \quad a = 46 \quad d = 5$$

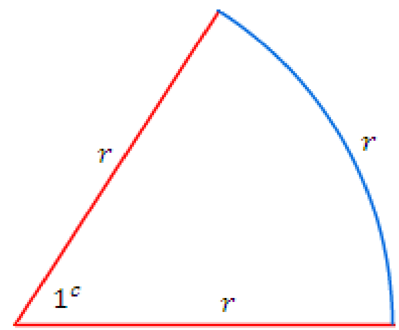
$$S_{11} = \frac{11}{2}\big(2(46) + 10(5)\big) = 781$$

Note: Generating the series, recognising that it is an arithmetic series and using the relevant formulae – is often the quickest method for cases where the expression within the summation is linear. For more complicated examples, at this stage you would only be dealing with relatively small numbers and could substitute in to work out every term.

Chapter 9 – Radian measure

Radians are an alternative method for measuring an angle, and are increasingly used in the advanced study of mathematics because of their many useful mathematical properties. Instead of being arbitrarily defined as $\frac{1}{360}$ of a full turn, as degrees are, radians are defined as follows:

The **angle** required for a sector of a circle so that the **arc length** is equal to the radius is defined as 1 radian (sometimes written as 1^c).



Since the arc is a portion of the circumference, it is related to the angle – in degrees – by the formula:

$$l = \frac{\theta}{360} 2\pi r$$

By defining radians in this way, 2π radians make a full turn, and the **formula for arc length** is:

$$l = \frac{\theta}{2\pi} 2\pi r \quad \textit{which simplifies to} \quad l = \theta r$$

$$\textit{Note that:} \quad l = r \quad \Longrightarrow \quad \theta = 1^c \qquad \textit{and} \qquad 1\ \textit{radian} \approx 57.3°$$

To **convert** between **degrees** and **radians**, it is simplest to consider a full turn:

$$360\ \textit{degrees} = 2\pi\ \textit{radians}$$

This can be used to find a **conversion factor** between degrees and radians:

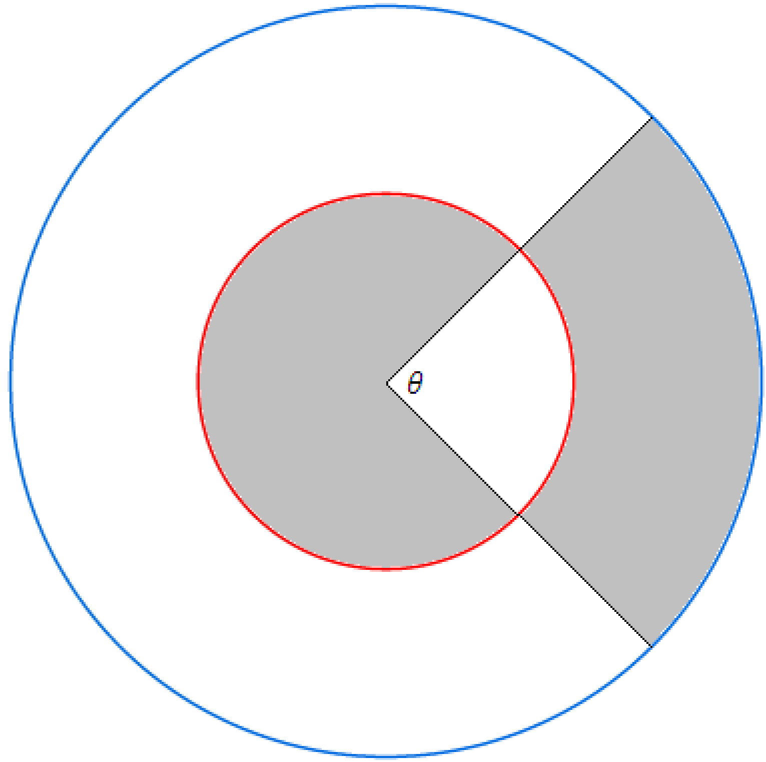
$$1\ \textit{degree} = \frac{\pi}{180}\ \textit{radians} \qquad \textit{or} \qquad 1\ \textit{radian} = \frac{180}{\pi}\ \textit{degrees}$$

The **area of a sector** of a circle is given by:

$$A = \pi r^2 \times \frac{\theta}{2\pi} \quad \textit{which simplifies to} \quad A = \frac{1}{2} r^2 \theta$$

Eg: The two concentric circles in the diagram are of radii $5cm$ and $10cm$ respectively.

Calculate the value of θ required to make the areas equal.



Calculate the area of the left-hand part:

$$A = \frac{1}{2}5^2(2\pi - \theta) = 25\pi - 12.5\theta$$

Calculate the area of the right-hand part:

$$A = \frac{1}{2}10^2\theta - \frac{1}{2}5^2\theta = 37.5\theta$$

Equate and solve:

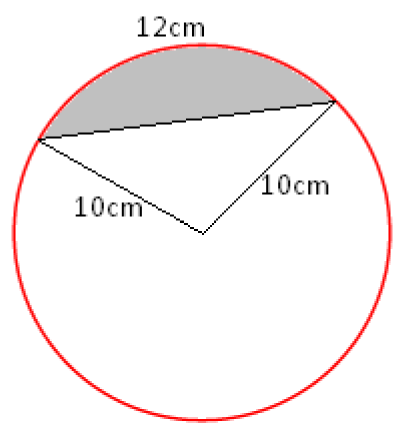
$$37.5\theta = 25\pi - 12.5\theta$$

$$50\theta = 25\pi$$

$$\theta = \frac{\pi}{2}$$

Note: You should also be confident finding the area of a segment using radians, incorporating content from chapter 4 to deal with circles.

Eg: Find the area of the segment with corresponding arc length $12cm$ on a circle of radius $10cm$.



Use the arc length formula to find the angle:

$$l = r\theta \quad \Longrightarrow \quad 12 = 10\theta \quad \Longrightarrow \quad \theta = 1.2\ radians$$

Find the area of the sector:

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}10^2(1.2) = 60$$

Find the area of the triangle using trigonometry:

$$A = \frac{1}{2}ab\sin C = \frac{1}{2}(10)(10)\sin(1.2) = 46.6cm^2\ to\ 1\ d.p.$$

Subtract the area of the triangle from that of the sector to find the area of the segment in question:

$$60 - 46.6 = 13.4cm^2\ to\ 1\ d.p.$$

Chapter 10 – Further trigonometry with radians

As the maths course progresses, radians become increasingly the unit of choice when dealing with angles, so it is important that you are familiar with the most common angles in both units:

Degrees ⟷ Radians

$0° = 0^c$	$90° = \frac{\pi^c}{2}$
$30° = \frac{\pi^c}{6}$	$180° = \pi^c$
$45° = \frac{\pi^c}{4}$	$270° = \frac{3\pi^c}{4}$
$60° = \frac{\pi^c}{3}$	$360° = 2\pi^c$

When you are dealing with **trigonometrical equations** from now on, you can expect to see intervals given in radians at least as often as degrees. Where the interval is quoted in radians (eg, in terms of π), your solutions should all be given in radians.

Eg: Find all solutions to the equation $\tan(2x - \frac{\pi}{2}) = \tan\left(\frac{2\pi}{3}\right)$ in the interval $-\pi \le x \le \pi$.

Convert the interval:

$$-\pi \le x \le \pi \quad \Longrightarrow \quad -\frac{5\pi}{2} \le 2x - \frac{\pi}{2} \le \frac{3\pi}{2}$$

Find a primary solution:

$$2x - \frac{\pi}{2} = \tan^{-1}\left(\tan\left(\frac{2\pi}{3}\right)\right) = \frac{2\pi}{3}$$

Use the graph to identify all solutions within the range:

$$2x - \frac{\pi}{2} = \frac{2\pi}{3} \quad and \quad -\frac{\pi}{3} \quad and \quad -\frac{4\pi}{3} \quad and \quad -\frac{7\pi}{3}$$

Rearrange to give all solutions for x:

$$x = \frac{7\pi}{12} \quad x = \frac{\pi}{12} \quad x = -\frac{5\pi}{12} \quad x = -\frac{11\pi}{12}$$

Note: Solutions, where possible, are usually left in terms of π, and as improper fractions where necessary. When decimal approximations are unavoidable, only round at the end to avoid compounding errors through subsequent working, and always round to a sufficiently high degree of accuracy (usually 2 decimal places or 3 significant figures is ample).

Chapter 11 – Exponentials and logarithms

An **exponential function** is a function where the variable appears as an **exponent**, or power.

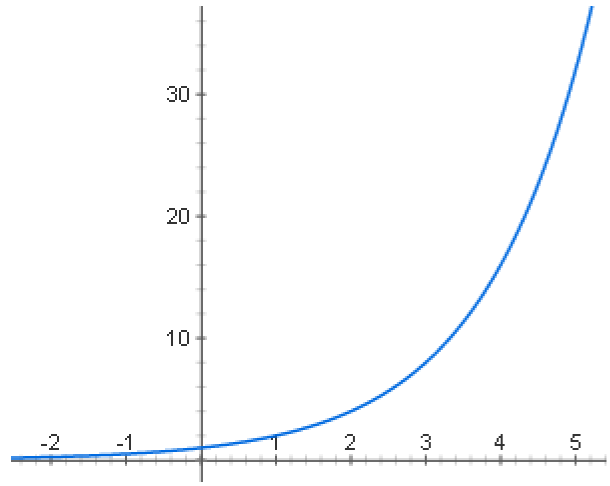
Eg: $y = 2^x$ is an exponential function with base 2.

Note that the graph has an asymptote at $y = 0$ (the x axis) because $2^x > 0$ for all values of x.

The graph cuts the y axis at $(0,1)$ because $2^0 = 1$.

For negative values of x, the graph gives y values less than 1.

Points on the graph include: $(10,1024)$, $(-3,0.125)$.



The **exponential graph** $y = a^x$, for $a > 0$, passes through the point $(0,1)$ because $a^0 = 1$

The **inverse function** of the exponential function is called the **logarithm**. It is defined as:

$$N = a^x \quad \Leftrightarrow \quad x = \log_a N$$

Note: Another name for the power (or index, or exponent) is *logarithm*. a is the base in both cases, and x, the exponent (or logarithm) is equal to the log of N.

$\log_a N$ is described as "**the logarithm of *N* to the base *a***", or "log to the base a, of N". If the base is irrelevant or can be assumed (see common bases later on), this is often just said as "log N".

The logarithm of a particular number can be thought of as the power to which the base must be raised in order to make that number.

Eg: $\log_{10} 1000 = 3$ since $10^3 = 1000$, and $\log_2 65536 = 16$ since $2^{16} = 65536$.

Since the logarithm function is the **inverse** of the exponential function, the laws governing manipulation of logs are direct results of the laws of indices (covered in C1):

$$a^0 = 1 \quad \Rightarrow \quad \log_a 1 = 0 \qquad\qquad a^1 = a \quad \Rightarrow \quad \log_a a = 1$$

$$a^n a^m = a^{n+m} \quad \Rightarrow \quad \log_a x + \log_a y = \log_a (xy) \qquad \frac{a^n}{a^m} = a^{n-m} \quad \Rightarrow \quad \log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$$

$$(a^n)^m = a^{nm} \quad \Rightarrow \quad k \log_a x = \log_a x^k$$

To **solve** an equation in the form $\boldsymbol{a^x = b}$, take logs of both sides, rearrange and simplify.

Eg: Moore's law suggests that the processing power of computers increases by 50% every year. How long will it be until computers are running at a million times today's speeds?

Use a 50% increase multiplier to form an equation:

$$1.5^n = 1000000$$

Solve by taking logs:

$$\log_{10} 1.5^n = \log_{10} 1000000$$

$$n \log_{10} 1.5 = 6$$

$$n = \frac{6}{\log_{10} 1.5} = 34.1$$

Note: This was solved using base 10 (a common base – often abbreviated as $\log_{10} x = \lg x$) but would work equally well with any other base. Another common base is base 2, and the 'natural log' is to the base e (where e is the irrational number approximately equal to 2.718). You will learn more of the natural log (often abbreviated as $\log_e x = \ln x$) if you go on to study C3.

In general:

$$a^x = b \quad \Rightarrow \quad x = \frac{\log_{10} b}{\log_{10} a} = \frac{\log_k b}{\log_k a} \quad k > 0$$

Since another way of writing the equation $a^x = b$ is $x = \log_a b$, we can extend this method to show:

$$\log_a b = \frac{\log_k b}{\log_k a}$$

Note: While a number of calculators may have a function allowing you to choose the base of your logarithm, by using this formula you can convert any log to a combination of logs with a more convenient base.

Note: You must become familiar with the key log rules and results above, as they will be essential in simplifying results. They can be difficult to learn since they are related to the inverse function of the exponential function (whose rules you should by now know well), and appear counter-intuitive at first simply because they work in the opposite direction to those of the exponential function.

Eg: Simplify the expression $\log_2 48 - 2\log_2 12$.

$$\log_2 48 - 2\log_2 12 = \log_2 48 - \log_2 144 = \log_2 \frac{48}{144} = \log_2 \frac{1}{3} = \log_2 3^{-1} = -\log_2 3$$

Chapter 12 – Geometric series

A **geometric series** is a series (the sum of a sequence of numbers) where each term is found by multiplying the previous term by a constant, known as the **common ratio**. It can be written as:

$$a + ar + ar^2 + ar^3 + \cdots$$
$$where\ a\ is\ the\ first\ term\ and\ r\ is\ the\ common\ ratio$$

The $\boldsymbol{n^{th}}$ **term of a geometric series** is given by:

$$U_n = ar^{n-1}$$

Note: This formula **is** in the formula book. It is included here for completeness. While you do not necessarily need to memorise it, it would be useful to understand where it comes from.

The **sum of the first $\boldsymbol{n}$ terms** of a geometric series is given by:

$$S_n = \frac{a(1-r^n)}{1-r}$$

Note: This formula **is** in the formula book. It is included here for completeness. Make sure you are confident applying it.

Eg: A geometric series has third term 36 and sixth term 972. Find the sum of the first 8 terms.

Construct simultaneous equations in a and r using the formula for the n^{th} term:

$$U_3 = ar^2 = 36 \qquad U_6 = ar^5 = 972$$

Solve simultaneously (these can be solved using elimination through division):

$$\frac{ar^5}{ar^2} = \frac{972}{36} \quad \Rightarrow \quad r^3 = 27 \quad \Rightarrow \quad r = 3 \quad \Rightarrow \quad a = \frac{36}{3^2} = 4$$

Use the sum formula to find the required sum:

$$S_8 = \frac{a(1-r^8)}{1-r} = \frac{4(1-3^8)}{1-3} = 13120$$

Note: If the common ratio is negative, terms will be positive and negative alternately.

Eg: Write the first four terms of any valid geometric series with fifth term 64 and seventh term 16.

$$U_5 = 64 \quad U_7 = 16 \quad \Rightarrow \quad \frac{ar^4}{ar^6} = \frac{64}{16} \quad \Rightarrow \quad r^2 = \frac{1}{4} \quad \Rightarrow \quad r = \pm\frac{1}{2} \quad \Rightarrow \quad a = 1024$$

$$1024 + 512 + 256 + 128 + \cdots \qquad or \qquad 1024 - 512 + 256 - 128 + \cdots$$

A geometric series **converges** when $|r| < 1$. This is equivalent to saying $-1 < r < 1$.

Note: This formula **is** in the formula book. It is included here for completeness.

A geometric series that **converges**, tends to a limit. The sum to infinity is given by:

$$S_\infty = \frac{a}{1-r}$$

Note: This formula **is** in the formula book. It is included here for completeness. It is worth noticing that this formula is obtainable from the formula for S_n by noting that, for $|r| < 1$, $r^n \to 0$ as $n \to \infty$.

Eg: Find the limit of the sum $99 + 9 + 0.9 + 0.09 + \cdots$.

$$a = 99 \quad r = 0.1 \quad \Longrightarrow \quad S_\infty = \frac{a}{1-r} = \frac{99}{1-0.1} = \frac{99}{0.9} = \frac{990}{9} = 110$$

Note: If you are required to work backwards, to find the number of terms for a geometric series, it may be necessary to use logarithms.

Eg: The sum of the first n terms of the geometric series $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$ is greater than 2.999. Find the minimum possible value for n.

$$S_n = \frac{a(1-r^n)}{1-r} \quad \Longrightarrow \quad 2.999 = \frac{1\left(1-\left(\frac{2}{3}\right)^n\right)}{\left(1-\frac{2}{3}\right)} = 3\left(1-\left(\frac{2}{3}\right)^n\right) \quad \Longrightarrow \quad \frac{2.999}{3} = 1-\left(\frac{2}{3}\right)^n$$

$$\left(\frac{2}{3}\right)^n = 1 - \frac{2.999}{3} = \frac{0.001}{3} \quad \Longrightarrow \quad \log\left(\frac{2}{3}\right)^n = \log\left(\frac{0.001}{3}\right) \quad \Longrightarrow \quad n\log\left(\frac{2}{3}\right) = \log\left(\frac{0.001}{3}\right)$$

$$n = \frac{\log\left(\frac{0.001}{3}\right)}{\log\left(\frac{2}{3}\right)} = 19.74\ldots \quad \Longrightarrow \quad n_{min} = 20$$