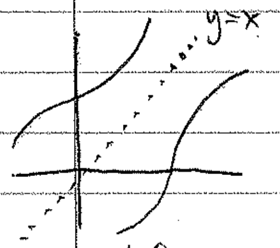
C3 Summary of Notes

Fractions:
- $\frac{a}{b} \pm \frac{c}{d} \equiv \frac{ad \pm bc}{bd}$
- $\frac{a}{b} + \frac{c}{b^2} \equiv \frac{ab + c}{b^2}$
- $\frac{a}{b} \times \frac{c}{d} \equiv \frac{ac}{bd}$
- $k \times \frac{a}{b} \equiv \frac{ka}{b}$
- $\frac{a}{b} \div \frac{c}{d} \equiv \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$
- $\frac{a}{b} \div k = \frac{a}{bk}$

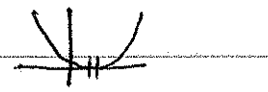
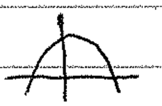
Functions:
- $y = f(x)$ for each value of x in the domain there is only one value of y in the range



(domain of $f \equiv$ range of f^{-1})

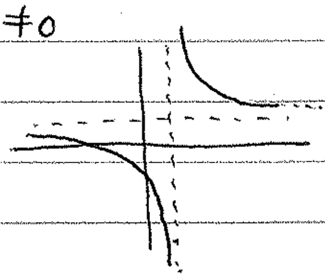
- composite functions: $fg(x) \equiv f(g(x))$
- inverse functions $ff^{-1}(x) \equiv x = f^{-1}f(x)$
- inverse only exists if f is 1-1 (increasing or decreasing)
- to find f^{-1}, rearrange to get x as the subject then exchange x for y and y for x.

Quadratic Functions $y = ax^2 + bx + c$

	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
$a > 0$			
$a < 0$			

- line of symmetry (x-value corresponding to max/min) occurs where $x = -b/2a = \frac{\alpha + \beta}{2}$
- $y = a(x+b)^2 + c$ has minimum at $(-b, c)$
  $y = c - a(x+b)^2$ has maximum at $(-b, c)$

Rectangular Hyperbola $y = \frac{ax+b}{cx+d}$ $c \neq 0$

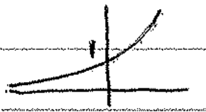


- root at $y=0$, $x = -b/a$
- intercept at $x=0$, $y = b/d$
- vertical asymptote at $x = -d/c$
- horizontal asymptote at $y = a/c$
- rotational symmetry about intersection of asymptotes

Natural Logarithms

- $y = e^x \Leftrightarrow x = \ln y$
- $e^{\ln x} \equiv x$ and $\ln e^x \equiv x$
- $\ln A + \ln B \equiv \ln AB$
- $\ln A - \ln B \equiv \ln (A/B)$
- $n \ln A = \ln A^n$
- $\ln(A \pm B)$ cannot be simplified
- $\log_a b \equiv \frac{\ln b}{\ln a}$

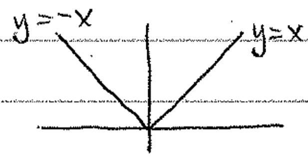
Exponential and Logarithmic Functions

$f(x) = e^x$, $x \in \mathbb{R}$  has range $y > 0$

the inverse $f^{-1}(x) = \ln x$, $x > 0$  has range $y \in \mathbb{R}$

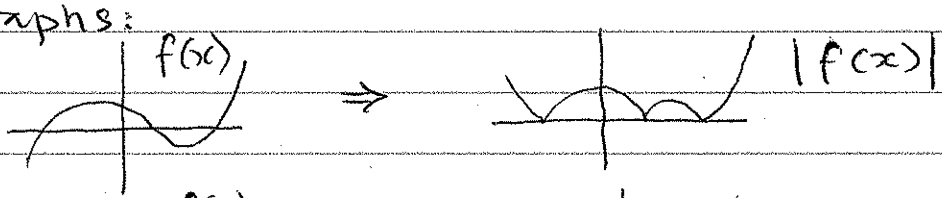
Modulus Functions

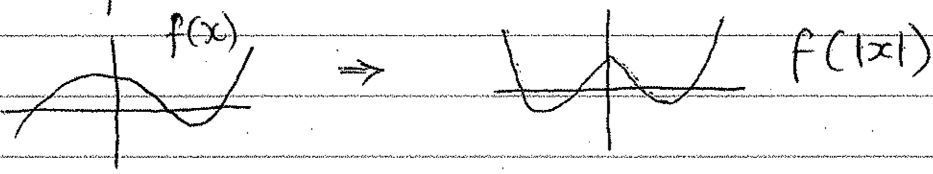
$$|x| \equiv \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



modulus equations
(i) $|f(x)| = k > 0$: solve $f(x) = \pm k$
(ii) $|f(x)| = |g(x)|$: solve $f(x) = \pm g(x)$
(iii) $|f(x)| = g(x)$: sketch graphs and solve suitable equations

modulus graphs:

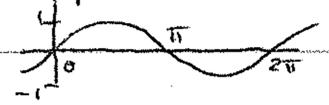
(i) 

(ii) 

Transformations:

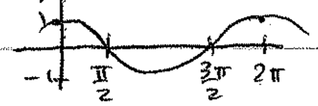
(i) $f(x+k)$ is a translation $\begin{pmatrix} -k \\ 0 \end{pmatrix}$
(ii) $f(x)+k$ is a translation $\begin{pmatrix} 0 \\ k \end{pmatrix}$
(iii) $f(kx)$ is enlargement by factor $\frac{1}{k}$ in x-direction
(iv) $k f(x)$ is enlargement by factor k in y-direction
(v) $f(-x)$ is reflection in y-axis
(vi) $-f(x)$ is reflection in x-axis
(vii) $f^{-1}(x)$ is reflection in $y = x$

<u>Trig Functions</u> • $f(x) \equiv \sin x$, $x \in \mathbb{R}$ has period 2π and range $-1 \le y \le 1$



$\sin(-x) \equiv -\sin x$

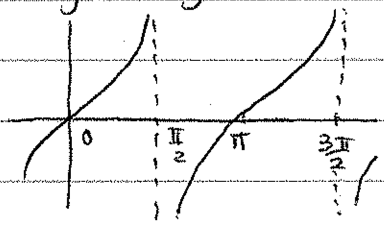
• $f(x) = \cos x$, $x \in \mathbb{R}$ has period 2π and range $-1 \le y \le 1$



$\cos(-x) \equiv \cos x$

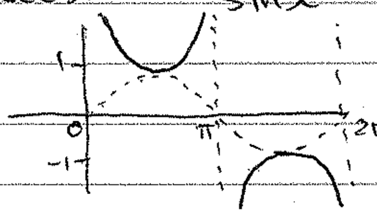
• $f(x) = \tan x$, $x \in \mathbb{R}$, $x \neq (2n+1)\frac{\pi}{2}$ has period π and range $y \in \mathbb{R}$

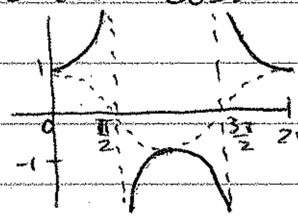
$\tan(-x) \equiv -\tan x$

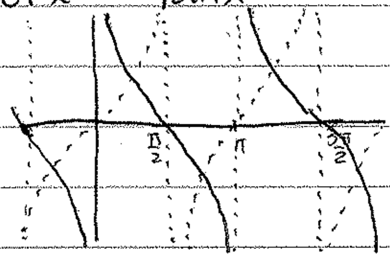


<u>Minor trig Functions</u>

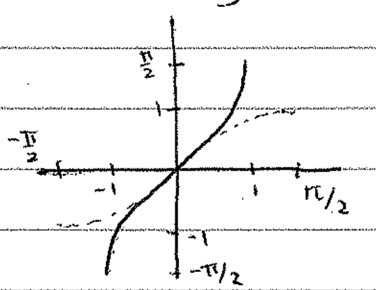
$\operatorname{cosec} x \equiv \frac{1}{\sin x}$ $\quad \sec x \equiv \frac{1}{\cos x}$ $\quad \cot x \equiv \frac{1}{\tan x}$

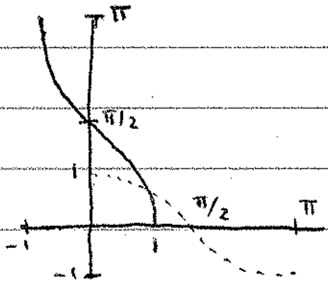


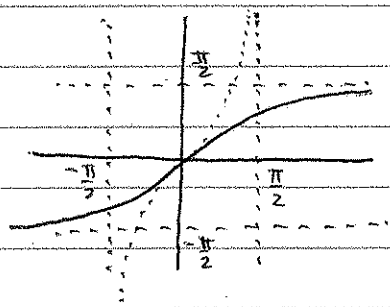




<u>Inverse trig functions</u>







$f(x) = \arcsin x$ has domain $-1 \le x \le 1$ and range $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$f(x) = \arccos x$ has domain $-1 \le x \le 1$ and range $0 \le y \le \pi$

$f(x) = \arctan x$ has domain $x \in \mathbb{R}$ and range $-\frac{\pi}{2} < y < \frac{\pi}{2}$

<u>Trig Identities</u>

• $\tan x \equiv \frac{\sin x}{\cos x}$ $\quad$ • $\cot x \equiv \frac{\cos x}{\sin x}$

• complementary angles:

if $\alpha + \beta = \pi/2$, $\sin\alpha = \cos\beta$ and $\tan\alpha = \cot\beta$

- **Pythagorean Identities** $\sin^2 A + \cos^2 A \equiv 1$
  $1 + \tan^2 A \equiv \sec^2 A$
  $1 + \cot^2 A \equiv \operatorname{cosec}^2 A$

- **Compound Angles:** $\sin(A \pm B) \equiv \sin A \cos B \pm \cos A \sin B$
  $\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$
  $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

- **Double Angles:** • $\sin 2A \equiv 2 \sin A \cos A$
- $\tan 2A \equiv \dfrac{2 \tan A}{1 - \tan^2 A}$ • $\cos 2A \equiv \cos^2 A - \sin^2 A$
  $\equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$
  - replace A with $\frac{1}{2}\theta$ to get Half-Angle Formulas
  - rearrangements: $\cos^2 A \equiv \frac{1}{2}(1 + \cos 2A)$
    $\sin^2 A \equiv \frac{1}{2}(1 - \cos 2A)$

- **Compound Angle Transformations**
  $a \sin x \pm b \cos x \equiv R \sin(x \pm \alpha)$ $[R^2 = a^2 + b^2]$
  $a \cos x \pm b \sin x \equiv R \cos(x \mp \alpha)$

- **Addition Formulae** $2 \sin A \cos B \equiv \sin(A+B) + \sin(A-B)$
  [deduced from Compound Angle Formulae] $2 \cos A \sin B \equiv \sin(A+B) - \sin(A-B)$
  $2 \cos A \cos B \equiv \cos(A+B) + \cos(A-B)$
  $2 \sin A \sin B \equiv \cos(A-B) - \cos(A+B)$

- **Product Formulae** $\sin P + \sin Q \equiv 2 \sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
  [deduced from Addition Formulae] $\sin P - \sin Q \equiv 2 \cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$
  $\cos P + \cos Q \equiv 2 \cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$
  $\cos P - \cos Q \equiv -2 \sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$

- **General Solutions**
  $\sin x = \sin y \Rightarrow x = \begin{cases} y \\ \pi - y \end{cases} + n.2\pi$
  $\cos x = \cos y \Rightarrow x = \pm y + n2\pi$
  $\tan x = \tan y \Rightarrow x = y + n\pi$

Numerical Solutions of Equations [x is in radians]
- $f(x) = 0$ has a solution in the interval $a < x < b$ if $f(a)$ and $f(b)$ have opposite signs.
- $f(x) = 0$ is rearranged to get $x = g(x)$. The ITERATION formula $x_{n+1} = g(x_n)$ converges to a root of $f(x) = 0$ as $n \to \infty$ (or fails)

Calculus [x is in radians]

function	derivative
y	dy/dx
$f(x)$	$f'(x)$
x^n	nx^{n-1}
e^x	e^x
$\ln x$	$1/x$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\operatorname{cosec}^2 x$

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

Chain Rule:
- $y = f(g(x)) \Rightarrow \frac{dy}{dx} = f'(g(x)) \cdot g'(x)$
- $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Special cases:
- $y = f(ax+b) \Rightarrow \frac{dy}{dx} = a \cdot f'(ax+b)$
- $y = e^{f(x)} \Rightarrow \frac{dy}{dx} = f'(x) \cdot e^{f(x)}$
- $y = \ln(f(x)) \Rightarrow \frac{dy}{dx} = f'(x)/f(x)$
- $y = \sin^n x = (\sin x)^n \Rightarrow \frac{dy}{dx} = n(\sin x)^{n-1} \cdot \cos x$
- $y = \cos^n x = (\cos x)^n \Rightarrow \frac{dy}{dx} = -n(\cos x)^{n-1} \sin x$

Product Rule $y = uv \Rightarrow \frac{dy}{dx} = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}$

Quotient Rule $y = \frac{u}{v} \Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$