OCR Core 3 Module Revision Sheet

The C3 exam is 1 hour 30 minutes long. You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted...always an area prone to error.

J.M.S.

Functions

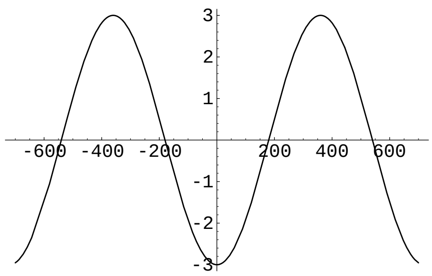
- A *function* is a *one-to-one* or a *many-to-one mapping*. There are also *many-to-many* and *one-to-many* mappings, but these are ***not*** functions. In a function, for every value you feed into the function you obtain one (and only one) value out.
- The *domain* of a function $y = f(x)$ is all the possible values of x the function can take. For example the domain of $y = \sqrt{x-4}$ is $x \geqslant 4$. In other words all the *inputs* the function can take.
- The *range* of a function is all the possible *outputs*. That is all the possible values of $f(x)$. So for $f(x) = -x^2 + 5$ the range is $f(x) \leqslant 5$.
- Functions are transformed as follows

Function	Graph Shape
$f(x)$	Normal Graph
$2f(x)$	Graph stretched by a factor of 2 parallel to the y-axis i.e. every value of $f(x)$ in the original graph is multiplied by 2
$f(2x)$	Graph stretched by factor of $\frac{1}{2}$ parallel to the x-axis
$3f(\frac{x}{4})$	Graph stretched by factor of 4 parallel to the x-axis and a stretch by a factor of 3 parallel to the y-axis
$f(x)+6$	Graph translated vertically *up* 6 units
$f(x)-6$	Graph translated vertically *down* 6 units
$f(x+4)$	Graph translated 4 units to the *left*
$f(x-6)$	Graph translated 6 units to the *right*
$f(x-6)+9$	Graph translated 6 units to the *right* and 9 units *up*. This is a translation and can be expressed as $\binom{6}{9}$ where $\binom{\text{change in } x}{\text{change in } y}$
$-f(x)$	Graph reflected in the x-axis
$f(-x)$	Graph reflected in the y-axis

- When faced with more than one of the above transformations it sometimes matters which order you carry out the transformations. In the example of $2f(x-3)$ it doesn't matter because you end up with the same result both ways, regardless of whether you do the translation right, or the stretch parallel to the y-axis first (think about it). However with $f(2x+10)$ you get a different result depending on the order you carry out the translation 10 left and then stretch factor $\frac{1}{2}$ parallel to the x-axis. If the conflict occurs within the bracket you should do the *opposite* of what you expect. So here you do the translation first and then the stretch.

  For $2f(x)+6$ the transformations are outside the bracket, so here you would do the stretch *then* the translation.

- So for example if you were asked to sketch $y = 3\sin(\frac{x}{2} - 90)$ you would translate '$y = \sin x$' 90° to the right, *then* stretch factor 2 parallel to the x-axis and stretch factor 3 parallel to the y-axis.



- If $f(x) = f(-x)$ then the function is called an *even* function. An even function is one where the y-axis is a line of symmetry. Examples are

$$f(x) = \cos x \qquad \text{since} \qquad f(-x) = \cos(-x) = \cos x = f(x),$$
$$g(x) = x^2 + 1 \qquad \text{since} \qquad g(-x) = (-x)^2 + 1 = x^2 + 1 = g(x).$$

- If $-f(x) = f(-x)$ then the function is called an *odd* function. An odd function is one where the function is unchanged if you rotate it 180° around the point $(0, 0)$. Examples are

$$f(x) = \sin x \qquad \text{since} \qquad f(-x) = \sin(-x) = -\sin x = -f(x),$$
$$g(x) = x^3 \qquad \text{since} \qquad g(-x) = (-x)^3 = -x^3 = -g(x).$$

- You must be able to construct compositions of functions. Note that $f(g(x))$ is not usually the same as $g(f(x))$. For example if $f(x) = x^2$ and $g(x) = x+1$ then $f(g(x)) = f(x+1) = (x+1)^2 = x^2 + 2x + 1$. Contrast this with $g(f(x)) = g(x^2) = x^2 + 1$.

- Sometimes you will be asked to describe a quadratic of the form $ax^2 + bx + c$ in terms of $f(x) = x^2$. It is often useful to *complete the square*. Very quickly I will go through a couple of examples of how to do this:

$x^2 + 10 \Rightarrow$ Clearly just $f(x) + 10$.

$x^2 + 6x + 10 \Rightarrow$ Complete square to get $(x+3)^2 - 9 + 10 = (x+3)^2 + 1$ so it is $f(x+3) + 1$, which is the translation $\binom{-3}{1}$ of x^2.

$2x^2 + 16x + 1 \Rightarrow$ Complete square to get $2(x+4)^2 - 31$ so it is $2f(x+4) - 31$, which is a stretch of factor 2 away from the x-axis, followed by a translation $\binom{-4}{-31}$ of x^2.

- The inverse of a function $f(x)$ is denoted $f^{-1}(x)$. To find the inverse of a function you swap round the x and the y and make y the subject again. This will be the inverse of the original function. For example find the inverse of $f(x) = \sqrt{x^3 + 2}$ gives

$$\begin{aligned} f(x) &= \sqrt{x^3 + 2}, \\ \Rightarrow \qquad y &= \sqrt{x^3 + 2}, \\ \Rightarrow \qquad x &= \sqrt{y^3 + 2}, \\ \Rightarrow \qquad y &= \sqrt[3]{x^2 - 2}, \\ \Rightarrow \qquad f^{-1}(x) &= \sqrt[3]{x^2 - 2}. \end{aligned}$$

- A function only has an inverse if it is a one-to-one mapping. If the original function is a many-to-one function (e.g. $y = x^2$ or any of the trig functions) you must restrict its domain to make it a one-to-one mapping (e.g. for $y = x^2$ restrict domain to $x \geqslant 0$). The domain and range of a function are switched in its inverse. For example if $f(x)$ has domain $x > 8$ and range $f(x) \leqslant -10$, then its inverse $f^{-1}(x)$ has domain $x \leqslant -10$ and range $f^{-1}(x) > 8$.

- Geometrically the relationship between a function and its inverse is a reflection in the line $y = x$. A useful spin-off from this result is that if you are asked to find where a function equals its inverse (i.e. $f(x) = f^{-1}(x)$) all you need to do is solve $f(x) = x$ or $f^{-1}(x) = x$; take your pick.

- Given a point on a function ($(3, 4)$, say) then the equivalent point on its inverse is $(4, 3)$ because it has been reflected in $y = x$. If the gradient at $(3, 4)$ was 7, then the gradient on the inverse will be its reciprocal $\frac{1}{7}$.

Modulus

- The modulus function makes everything you put into it positive. For example $|4| = 4$ and $|-6| = 6$. If something negative is 'fed in' to the mod function then it multiplies it by -1 to turn it positive; otherwise it leaves it alone.

- If you have an expression such as $|x - 4|$, then the critical value for x is $x = 4$; if $x > 4$ then the expression is just $x - 4$ and if $x < 4$ then the expression becomes $-x + 4$ because the mod function multiplies it by -1 to turn it positive. This idea helps us solve modulus equations; for example to solve $|2x - 1| = 6$ we first look for the critical values of x; here clearly $x = \frac{1}{2}$. We therefore set up two equations depending on whether $x > \frac{1}{2}$ or $x < \frac{1}{2}$:

$$\begin{aligned} \text{If } x &< \tfrac{1}{2} \\ \text{then } -2x + 1 &= 6 \\ x &= -\tfrac{5}{2}, \end{aligned} \qquad\qquad \begin{aligned} \text{If } x &> \tfrac{1}{2} \\ \text{then } 2x - 1 &= 6 \\ x &= \tfrac{7}{2}. \end{aligned}$$

  We perform a little check at the end to check that the solutions found actually satisfy the conditions on x are met; the left hand equation is valid if $x < \frac{1}{2}$ and the solution we have found *is* less that $\frac{1}{2}$; the right hand equation is valid if $x > \frac{1}{2}$ and the solution *is* greater than $\frac{1}{2}$. Therefore both solutions found are valid.

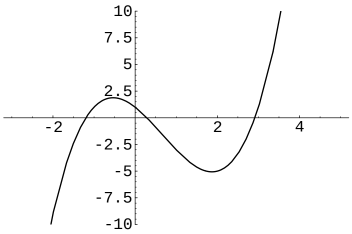
- If you merely have an equation such as $|\text{something}| = |\text{something else}|$ then just get rid of the mods and square both sides to get $(\text{something})^2 = (\text{something else})^2$. *Check* your answers back in the original mod equation to check they work!

- Consider the intimidating looking $|2x - 1| - 1 < |x + 2|$. As with most inequalities a good first step is to solve the *equality*; i.e. solve $|2x - 1| - 1 = |x + 2|$. The critical x values are $x = -2$ and $x = \frac{1}{2}$ so we need to set up three different equations depending whether x is $x < -2$, $-2 < x < \frac{1}{2}$ or $x > \frac{1}{2}$ and solve:

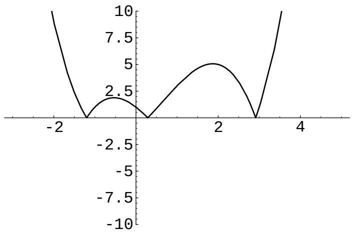
$$\begin{aligned} \text{If } x &< -2 \\ \text{then } -2x + 1 - 1 &= -x - 2 \\ x &= 2, \end{aligned} \qquad \begin{aligned} \text{If } -2 < x &< \tfrac{1}{2} \\ \text{then } -2x + 1 - 1 &= x + 2 \\ x &= -\tfrac{2}{3}, \end{aligned} \qquad \begin{aligned} \text{If } x &> \tfrac{1}{2} \\ \text{then } 2x - 1 - 1 &= x + 2 \\ x &= 4. \end{aligned}$$

  Performing our check again we see that two solutions are fine, but $x = 2$ is *not* a solution because the equation was only valid if $x < -2$. Therefore the solution of the equation is $x = -\frac{2}{3}$ or $x = 4$. To solve the inequality we need to see if a number less than $-\frac{2}{3}$ works in the inequality (it doesn't), to see if a number between $-\frac{2}{3}$ and 4 works (it does) and to see if a number greater than 4 works (it doesn't). Therefore $-\frac{2}{3} < x < 4$ is the solution to the question.

- Given a graph of $y = f(x)$ you must be able to draw the graph of $y = |f(x)|$; this is done by leaving any parts of the curve above the x-axis where they are and reflecting parts of the curve under the x-axis so that the are above the x-axis. In the reflected parts, the equation of the curve would be $y = -f(x)$. For example:



goes to



.

Trigonometry

- By definition $\sec\theta \equiv \dfrac{1}{\cos\theta}$, $\quad \operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$, $\quad \cot\theta \equiv \dfrac{1}{\tan\theta}$.
- If you get an equation where one of the new trig functions equals a constant, then just take the reciprocal of each side and solve *à la* C2. For example

$$\sec\theta = 5 \qquad \Rightarrow \qquad \frac{1}{\cos\theta} = 5 \qquad \Rightarrow \qquad \cos\theta = \frac{1}{5}.$$

- Know the graphs of $y = \sec x$, $y = \operatorname{cosec} x$ and $y = \cot x$. Page 91/2 of your textbook.
- By dividing $\sin^2 x + \cos^2 x \equiv 1$ by $\sin^2 x$ and $\cos^2 x$ we can derive

$$1 + \cot^2 x \equiv \operatorname{cosec}^2 x \quad \text{and} \quad \tan^2 x + 1 \equiv \sec^2 x \quad \text{respectively.}$$

These create a whole new family of equations that reduce to a quadratic in disguise. For example solve $3\cot^2\theta + 5\operatorname{cosec}\theta + 1 = 0$ in the range $0 \leqslant \theta \leqslant 2\pi$. Firstly note we will need to replace the $\cot^2\theta$ by $\operatorname{cosec}^2\theta - 1$ to reduce the equation to one trig function only.

$$\begin{aligned} 3\cot^2\theta + 5\operatorname{cosec}\theta + 1 &= 0 \\ 3(\operatorname{cosec}^2\theta - 1) + 5\operatorname{cosec}\theta + 1 &= 0 \\ 3\operatorname{cosec}^2\theta + 5\operatorname{cosec}\theta - 2 &= 0 \\ (3\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 2) &= 0 \\ \operatorname{cosec}\theta = \tfrac{1}{3} \quad \text{or} \quad \operatorname{cosec}\theta &= -2. \end{aligned}$$

Therefore $\sin\theta = 3$ which has no solutions, or $\sin\theta = -\frac{1}{2}$ which gives the solutions $\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$.

- You must know, and be able to apply, the compound angle formulae:

$$\begin{aligned} \sin(A \pm B) &= \sin A\cos B \pm \cos A\sin B, \\ \cos(A \pm B) &= \cos A\cos B \mp \sin A\sin B, \\ \tan(A \pm B) &= \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}. \end{aligned}$$

- You must know, and be able to apply, the double angle formulae (derived by setting $A = B$ in the compound angle formulae above):

$$\begin{aligned} \sin 2A &= 2\sin A\cos A, \\ \cos 2A &= \cos^2 A - \sin^2 A, \\ &= 2\cos^2 A - 1, \\ &= 1 - 2\sin^2 A, \\ \tan 2A &= \frac{2\tan A}{1-\tan^2 A}. \end{aligned}$$

  Notice there are three versions of the double angle formula for $\cos 2A$; you need to *think hard* about which form you will need for the question you are solving. You will hardly ever need the fist of the three ($\cos^2\theta - \sin^2\theta$) because it involves two different trig functions; the aim is, usually, to get only one.

- You must be able to convert form the form $a\cos\theta \pm b\sin\theta$ into either $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$; the question will specify which. This then enables us to solve equations of the form

$$a\cos\theta \pm b\sin\theta = \text{constant}.$$

  For example express $3\cos\theta - 5\sin\theta$ in the form $R\cos(\theta + \alpha)$. Always start by looking at the coefficients of $\cos\theta$ and $\sin\theta$ in the original expression; here they are 3 and 5 (ignore the sign). Sum their squares and square root (like Pythagoras) and factorise out:

$$3\cos\theta - 5\sin\theta \equiv \sqrt{34}\left[\frac{3}{\sqrt{34}}\cos\theta - \frac{5}{\sqrt{34}}\sin\theta\right].$$

  Next consider the form of the answer we are aiming for; here "$R\cos(\theta+\alpha)$". The expansion of "$R\cos(\theta + \alpha)$" is "$R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$". Comparing

$$\sqrt{34}\left[\frac{3}{\sqrt{34}}\cos\theta - \frac{5}{\sqrt{34}}\sin\theta\right] \quad \text{with} \quad R(\cos\theta\cos\alpha - \sin\theta\sin\alpha)$$

  we see instantly $R = \sqrt{34}$. We also require $\frac{3}{\sqrt{34}} = \cos\alpha$ and $\frac{5}{\sqrt{34}} = \sin\alpha$; solving either of those two we find $\alpha = 59.0°$ (to 1 d.p). Therefore

$$3\cos\theta - 5\sin\theta \equiv \sqrt{34}\cos(\theta + 59.0°).$$

- The trig functions all have inverses if we restrict the domain. The conventional restrictions to allow inversion are

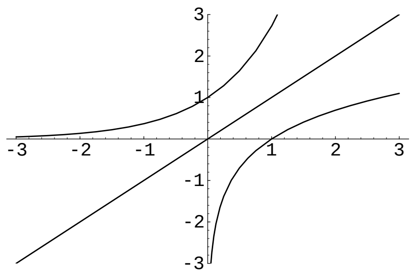
Function	Domain	Domain
$y = \sin x$	$-90° \leqslant x \leqslant 90°$	$-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$
$y = \cos x$	$0° \leqslant x \leqslant 180°$	$0 \leqslant x \leqslant \pi$
$y = \tan x$	$-90° < x < 90°$	$-\frac{\pi}{2} < x < \frac{\pi}{2}$

  Know what the graphs of $y = \sin^{-1} x$, $y = \cos^{-1} x$ and $y = \tan^{-1} x$ look like.

Exponentials & Logarithms

- Know that e is a special number in mathematics. It is approximately 2.7182818284... and it is irrational (i.e. it can't be expressed as a fraction; similar to π).

- If the base of a logarithm is e then we call it a 'natural logarithm'. Written $\log_e x \equiv \ln x$.
- We already know that logarithms and exponentials are inverses of each other with the relationships
$$\log_{10}(10^x) \equiv x \qquad \text{and} \qquad 10^{\log_{10} x} \equiv x.$$
The same is true for natural logarithms and exponents of e;
$$\ln(e^x) \equiv x \qquad \text{and} \qquad e^{\ln x} \equiv x.$$
- Below is a graph of $y = e^x$ and $y = \ln x$ showing the inverse relationship between the two (reflecting in $y = x$):



This also shows that you can't 'ln' a negative number and that $\ln 1 = 0$.
- All the laws of logarithms from C2 are true for natural logarithms (e.g. $\ln ab = \ln a + \ln b$). For example make a the subject of the following equation (a few steps missed out):
$$\begin{aligned} \ln(a-1) - \ln(a+1) &= b \\ \ln\left(\frac{a-1}{a+1}\right) &= b \\ \frac{a-1}{a+1} &= e^b \\ a(1-e^b) &= 1+e^b \\ a &= \frac{1+e^b}{1-e^b}. \end{aligned}$$
- You must understand that many physical systems can be modelled by either exponential growth or exponential decay. The most general form is $y = a \times b^x$. If $b > 1$ then the curve represents *exponential growth.* If $b < 1$ then the curve represents *exponential decay.* For example if the number of swine flu sufferers is modelled by $N = 5 \times 7^t$, where t is time measured in days, then find the amount of time for 2 billion people to have caught the disease. We need to solve $2 \times 10^9 = 5 \times 7^t$. So
$$\frac{2 \times 10^9}{5} = 7^t \quad \Rightarrow \quad \log\left(\frac{2 \times 10^9}{5}\right) = t \log 7 \quad \Rightarrow \quad t = 10.2 \text{ days! (to 3 s.f.).}$$
(Cue dramatic music...)
- Any exponential relationship $y = a \times b^x$ can be converted to an exponential form using e. This is useful because to differentiate exponential relationships they have to be of the form $y = a \times e^{kt}$. This is done using the powerful statement that (something $\equiv e^{\ln \text{something}}$), so
$$\begin{aligned} y &= a \times b^x \\ y &= a \times e^{\ln(b^x)} \\ y &= a \times e^{x \ln b} && \text{(by 'log law' } \log(a^n) = n \log a) \\ y &= a \times e^{kx}, && \text{(where } k = \ln b). \end{aligned}$$

- An exponential can never equal zero (see graph above). Therefore if you have an equation with lots of exponential 'bits' that you can factorise out, then you are allowed to divide through (in a way that is forbidden with trig functions). For example if $2x^2e^{2x} + 3xe^{2x} - 2e^{2x} = 0$, factorise out the e^{2x} to get $e^{2x}(2x^2 + 3x - 2) = 0$. Divide by e^{2x} to get $2x^2 + 3x - 2 = 0$ which solves to $x = \frac{1}{2}$ or $x = -2$.

- To differentiate an exponential the basic building block is

$$y = e^x \qquad \Rightarrow \qquad \frac{dy}{dx} = e^x.$$

That is *why* 'e' is so important; it gives us the exponential that differentiates to itself. Combined with the chain/product/quotient rule (below) we can build on this starting point. (Some students think that if $y = e^x$, then $\frac{dy}{dx} = xe^{x-1}$. Do not be one of them! Exponentials are fundamentally different to polynomials.)

- To differentiate a natural logarithm the basic building block is

$$y = \ln x \qquad \Rightarrow \qquad \frac{dy}{dx} = \frac{1}{x}.$$

Again combined with the chain/product/quotient rule (below) we can build on this starting point.

Differentiation

- The basic building blocks for differentiation (that we know at present) are:

$$\begin{array}{ccc} y = ax^n & y = e^x & y = \ln x \\ \dfrac{dy}{dx} = anx^{n-1}, & \dfrac{dy}{dx} = e^x, & \dfrac{dy}{dx} = \dfrac{1}{x}. \end{array}$$

Also we know the idea that (for $f(x) \equiv f$, $g(x) \equiv g$ and $k =$ constant)

$$\frac{d}{dx}(f + g) = \frac{d}{dx}(f) + \frac{d}{dx}(g) \quad \text{and} \quad \frac{d}{dx}(kf) = k\frac{d}{dx}(f).$$

(In big-boy speak we say that $\frac{d}{dx}$ is a linear operator.)

- The *chain rule* is incredibly important! It states that

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

This seems obvious from the way that differentials are written, but remember that they should not be thought of as fractions.

- If a bit of a $y = \ldots$ is making the differentiation difficult, then ask yourself the question "would making the complicated bit u make it easier for me to deal with?" For example with $y = (2x - 5)^{20}$ the function would be considerably easier if $u = 2x - 5$ because y becomes $y = u^{20}$. Similarly with $y = e^{x^2+1}$ my life would be easier if $u = x^2 + 1$ because y would become $y = e^u$.

- It can be applied as follows to the example $y = (x^4 + x)^{10}$. Let $u = x^4 + x$, so

$$\begin{array}{cc} y = u^{10} & u = x^4 + x \\ \dfrac{dy}{du} = 10u^9 & \dfrac{du}{dx} = 4x^3 + 1. \end{array}$$

Therefore $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 10u^9 \times (4x^3 + 1) = 10(4x^3 + 1)(x^4 + x)^9$.

- The above method works all the time but it is a little slow. You will notice the general result that if $y = [f(x)]^n$ then $\frac{dy}{dx} = n[f(x)]^{n-1} \times f'(x)$. So we can just write down the answer to similar problems. For example if $y = (3x^2 + 1)^5$ then $\frac{dy}{dx} = 30x(3x^2 + 1)^4$.

- We can also combine the chain rule with exponentials and logarithms to gain the following important results:

$$\begin{aligned} \frac{d}{dx}(e^{ax}) &= ae^{ax} && \text{using } u = ax \\ \frac{d}{dx}(e^{f(x)}) &= f'(x)e^{f(x)} && \text{using } u = f(x) \\ \frac{d}{dx}(\ln ax) &= \frac{a}{ax} = \frac{1}{x} && \text{using } u = ax \\ \frac{d}{dx}(\ln f(x)) &= \frac{f'(x)}{f(x)} && \text{using } u = f(x). \end{aligned}$$

- The *product rule* states that when $y = u \times v$ (where u and v are functions of x) we can differentiate it using the product rule. It states that

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

For example if $y = x^2(x^3 - 1)^3$ then

$$\begin{aligned} \frac{dy}{dx} &= [2x \times (x^3 - 1)^3] + [x^2 \times 3(x^3 - 1)^2 \times 3x^2] \\ &= 2x(x^3 - 1)^3 + 9x^4(x^3 - 1)^2 \\ &= x(x^3 - 1)^2[2(x^3 - 1) + 9x^3] \\ &= x(x^3 - 1)^2(11x^3 - 2). \end{aligned}$$

- With the product rule you often end up with expressions such as

$$\frac{dy}{dx} = 2x^3(2x + 1)^{-4} - x^4(2x + 1)^{-5}.$$

When tidying these things up you must pull out (as always) *the lowest power* of any common elements even if they are negative or fractional; here we have x^3 and $(2x + 1)^{-5}$:

$$\begin{aligned} \frac{dy}{dx} &= 2x^3(2x + 1)^{-4} - x^4(2x + 1)^{-5} \\ &= x^3(2x + 1)^{-5}[2(2x + 1) - x] \\ &= \frac{x^3(3x + 2)}{(2x + 1)^5}. \end{aligned}$$

- Very similar to the product rule is the *quotient rule*. It is used for functions of the form $y = \frac{u}{v}$. It states

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

For example differentiating $y = \frac{x^3}{x^2+1}$ gives

$$\frac{dy}{dx} = \frac{(x^2 + 1) \times 3x^2 - x^3 \times 2x}{(x^2 + 1)^2} = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}.$$

- Once again, although $\frac{dy}{dx}$ is not a fraction, it can be treated as such when taking its reciprocal, so

$$\frac{dy}{dx} = \frac{1}{\left(\frac{dx}{dy}\right)}.$$

For example if you have $V = \frac{4}{3}\pi r^3$ then $\frac{dV}{dr} = 4\pi r^2$ and also $\frac{dr}{dV} = \frac{1}{4\pi r^2}$. This idea most useful in the topic of...

- ... *connected rates of change*. Here you need to use the chain rule to 'connect' differentials you know to get one you need. Questions mostly ask you for $\frac{dy}{dx}$ (say) and you need to find a third variable to construct $\frac{dy}{dx} = \frac{dy}{d...} \times \frac{d...}{dx}$ by the chain rule. For example: The area A of a circle is increasing a rate of 3cm^2/s, find the rate at which the radius r is increasing when $r = 20$cm. We want to find $\frac{dr}{dt}$ so

$$\begin{aligned}\frac{dr}{dt} &= \frac{dr}{dA} \times \frac{dA}{dt} \qquad \text{but} \qquad A = \pi r^2, \text{ so } \frac{dA}{dr} = 2\pi r.\\ &= \frac{1}{2\pi r} \times 3\\ &= \frac{3}{40\pi}.\end{aligned}$$

Integration

- The central idea of calculus is that integration and differentiation are the inverse operations of each other in the same way that plus is the inverse operation of subtraction. In C3 a favourite type of question is to differentiate something using the above rules and then integrate something similar later in the question. View the question as a whole!

- Our basic building blocks for integration are therefore

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c, \qquad \int e^x\,dx = e^x + c, \qquad \int \frac{1}{x}\,dx = \ln x + c.$$

- A big result is gained by inspection below, but worth stating alone:

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c.$$

- Integration by *inspection* is effectively "spotting the answer" by an intermediate guess. The intermediate guess is then differentiated mentally and the final answer should then only be a constant factor out. Things to note are that the power on $e^{\text{something}}$ never changes and be on the lookout for integrals where the top line is *almost* the derivative of the bottom line. Here are a few examples:

Integral	Guess	Answer
$\int (2x+3)^{15}\,dx$	$(2x+3)^{16}+c$	$\frac{1}{32}(2x+3)^{16}+c,$
$\int (1-3x)^{-4}\,dx$	$(1-3x)^{-3}+c$	$\frac{1}{9}(1-3x)^{-3}+c,$
$\int 2\sqrt{4x+1}\,dx$	$(4x+1)^{\frac{3}{2}}+c$	$\frac{1}{3}(4x+1)^{\frac{3}{2}}+c,$
$\int 2e^{3x-5}\,dx$	$e^{3x-5}+c$	$\frac{2}{3}e^{3x-5}+c,$
$\int 7xe^{x^2+1}\,dx$	$e^{x^2+1}+c$	$\frac{7}{2}e^{x^2+1}+c,$
$\int \frac{7}{1-4x}\,dx$	$\ln(1-4x)+c$	$-\frac{7}{4}\ln(1-4x)+c,$
$\int \frac{e^{2x}}{1-e^{2x}}\,dx$	$\ln(1-e^{2x})+c$	$-\frac{1}{2}\ln(1-e^{2x})+c.$

You must practice this a lot... it only comes easily after a while. [Since most students also take C3 and C4 at the same time it is worth noting that all the above can be done by the C4 technique of *integration by substitution*.]

- $\int_a^b \pi y^2\,dx$ is the volume of revolution of the curve y rotated about the x-axis between $x=a$ and $x=b$. All that is needed for you to do is calculate y^2 in terms of x from y. For example find the volume of revolution of the solid formed by rotating the curve $y=\sqrt{2x+3}$ about the x-axis between $x=10$ and $x=14$. We need to evaluate $\int_a^b \pi y^2\,dx = \int_{10}^{14} \pi y^2\,dx$. Now the curve is $y=\sqrt{2x+3}$ so to find y^2 in terms of x we need only square the equation $\Rightarrow y^2 = 2x+3$. We therefore evaluate

$$\int_{10}^{14} \pi y^2\,dx = \pi \int_{10}^{14} (2x+3)\,dx = \pi\left[x^2+3x\right]_{10}^{14} = 108\pi.$$

- For volumes of revolution around the y-axis switch the x and the y and use $\int_p^q \pi x^2\,dy$ between $y=p$ and $y=q$. For example find the volume of revolution of the solid formed by rotating the line $y=3x-2$ about the y-axis between $y=0$ and $y=5$. We need to evaluate $\int_p^q \pi x^2\,dy = \int_0^5 \pi x^2\,dy$. Now the line is $y=3x-2$ so to find x^2 in terms of y, we make x the subject and square;

$$y=3x-2 \quad\Rightarrow\quad x=\frac{y+2}{3} \quad\Rightarrow\quad x^2=\frac{y^2+4y+4}{9}.$$

We therefore evaluate

$$\int_0^5 \pi x^2\,dy = \pi\int_0^5 \left(\frac{y^2+4y+4}{9}\right)dy = \frac{\pi}{9}\left[\frac{y^3}{3}+2y^2+4y\right]_0^5 = \frac{335\pi}{27}.$$

Numerical Methods

- Given an equation $f(x)=g(x)$ it is often not possible to solve them *analytically* (by algebraic manipulation) and we are forced to use numerical methods that 'home in' on the solution. You need to know two for C3: "search for a change of sign" and "fixed point iteration".

- *Search for a change of sign* 'homes in' on a solution to an equation by sandwiching the solution between two numbers. Those two numbers can gradually be brought together to improve knowledge of where the solution is. Given an equation ($e^x = 15x+3$, say) it is best to get one side equal to zero ($0 = e^x - 15x - 3$). Then *define* $f(x) = e^x - 15x - 3$. Then put values into $f(x)$ and look for a change of sign.

$$
\begin{aligned}
f(-1) &= 12.36787944\ldots && +\text{ve}\\
f(0) &= -2 && -\text{ve}\\
f(1) &= -15.28171817\ldots && -\text{ve}\\
f(2) &= -25.6109439\ldots && -\text{ve}\\
f(3) &= -27.91446308\ldots && -\text{ve}\\
f(4) &= -8.401849967\ldots && -\text{ve}\\
f(5) &= 70.4131591\ldots && +\text{ve}
\end{aligned}
$$

From this we can see that there are two solutions (α and β) such that

$$-1 < \alpha < 0 \qquad \text{and} \qquad 4 < \beta < 5.$$

If you were interested in finding β to 2 decimal places (say) then you would next evaluate

$$f(4.1), f(4.2), \ldots, f(4.9)$$

and you should discover $4.1 < \beta < 4.2$. Next

$$f(4.11), f(4.12), \ldots, f(4.19)$$

and you should discover $4.18 < \beta < 4.19$. You should resist the temptation (however strong) to state $\beta = 4.18$ (to 2 d.p.) as your final answer. It is still possible that the answer could still be $\beta = 4.19$ (to 2 d.p.). You must check 4.185 and then think! *Hard*!

We find $f(4.185) < 0$, so the change of sign exists between 4.185 and 4.19 so final stated answer should be $\beta = 4.19$ (to 2 d.p.)

- *Fixed point iteration* works by taking an equation and rearranging to isolate an x in the form $x = g(x)$. From this rearrangement we form an iterative formula

  $$x_{n+1} = g(x_n).$$

  It is important to note that there exist many possible rearrangements of an equation; for the equation $x^3 - 3x + 4 = 0$ here are a few:

  $$x = \sqrt[3]{3x-4} \qquad x = \frac{x^3+4}{3} \qquad x = \frac{3x-4}{x^2}.$$

  However, the exam will usually specify which one they want[1]. In the above example let's use the first one and create $x_{n+1} = \sqrt[3]{3x_n - 4}$. The starting value for the iteration is denoted x_0 (or x_1) and you should either use the value specified in the question or choose a value close to where you know the solution exists[2]. Here let's use $x_0 = -1$.

  To save time, you can use your calculator to speed up the process a lot. First type "$-1 =$" to enter -1 as the "Ans" on your calculator. Then type "$\sqrt[3]{(3 \times \text{Ans} - 4)}$". Press "="

[1] *rant* It is worth noting just how unrealistic this situation is; in practice you will discover that some of these rearrangements work a treat and some of them fail miserably. This should be part of a coursework (*à la* MEI) and not part of an exam! *rant*

[2] If the first part of a question gets you to show the solution exists between 2.1 and 2.2 then start with $x_0 = 2.1$

repeatedly to see the results of the iteration. You should find:

$$\begin{aligned}
x_0 &= -1 \\
x_1 &= -1.912931183 \\
x_2 &= -2.134410543 \\
x_3 &= -2.18324263 \\
x_4 &= -2.19321102 \\
x_5 &= -2.19528142 \\
&\vdots \qquad \cdots \text{keep pressing "=" lots and eventually} \cdots \\
x &= -2.19582 \text{ to (5 d.p.)}
\end{aligned}$$

Always state the accuracy to which you give your answer (sig figs or d.p.s). If when you keep pressing "=" it settles to one number we say the iteration *converges*; otherwise it *diverges*.

- Sometimes a question gives you an iteration and asks for the equation which has been solved (or to show that the number the iteration converges to represents a solution of another given equation). All you do is remove the $n+1$ and n subscripts and rearrange: For example

$$\begin{aligned}
x_{n+1} &= \ln(\sqrt[3]{1-2x_n}) \\
x &= \ln(\sqrt[3]{1-2x}) \\
e^x &= \sqrt[3]{1-2x} \\
e^{3x} + 2x - 1 &= 0.
\end{aligned}$$

Simpson's Rule

- Similar to the trapezium rule is *Simpson's Rule*. It can be used to approximate integrals. It uses a quadratic curve to approximate the curve rather than a straight line, and is therefore rather more accurate. Unlike the trapezium rule it is hard to say whether the approximation will be an over or under-estimate. Therefore you don't get questions on it.

- In class I refer to "Simpson Chunks"; this is a Stone-ism you will not hear elsewhere. One "Simpson Chunk" contains two intervals/strips and three ordinates.

  In general

  n "Simpson Chunks" $\Leftrightarrow$ $2n$ Intervals/Strips $\Leftrightarrow$ $2n+1$ Ordinates.

  The heights on the ordinates are the y-values of the curve. They are labelled y_0, y_1, ..., y_{2n}. **Never** forget that the first height on the left is denoted y_0 and **not** y_1; if you do the whole question will go wrong because your 'odds' and 'evens' will be wrong!

- Simpson's Rule states (where h is the distance between each ordinate/height):

$$\int_a^b y\,dx \approx \frac{h}{3}\left[y_0 + y_{2n} + 4(y_1 + y_3 + \cdots + y_{2n-1}) + 2(y_2 + y_4 + \cdots + y_{2n-2})\right].$$

$$\int_a^b y\,dx \approx \frac{h}{3}\left[\text{'sum ends'} + 4(\text{'sum internal odds'}) + 2(\text{'sum internal evens'})\right].$$

- For example use 8 intervals to approximate $\int_{-4}^{4} \frac{1}{1+x^2}\,dx$. Each interval must have width 1 since the total width is 8. There must be nine ordinates. A table for the ordinates:

y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
$\frac{1}{17}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{17}$

  Therefore

$$\begin{aligned}\int_{-4}^{4} \frac{1}{1+x^2}\,dx &\approx \frac{1}{3}\left[\frac{1}{17}+\frac{1}{17}+4\left(\frac{1}{10}+\frac{1}{2}+\frac{1}{2}+\frac{1}{10}\right)+2\left(\frac{1}{5}+1+\frac{1}{5}\right)\right]\\ &\approx 2.573 \text{ (to 3 d.p.)}.\end{aligned}$$