MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

C3, Methods for Advanced Mathematics

(Version B—reference to new book)

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References: Chapter 1 Pages 2-3

Exercise 1A Q. 2

Proof by direct argument

Some proofs can be constructed using known facts (e.g. the square of an even number is even.)

Geometric proofs can be constructed by direct argument.

E.g. Prove that a number is divisible by 3 if the sum of digits is divisible by 3.

Let the number be $x = a + 10b + 100c + ...$

The sum of digits is $s = a + b + c + ...$

$x - s = 9b + 99c + 999d + ...$

$\Rightarrow x = s + 3(3b + 33c + ...)$

So if s is divisible by 3 then so is the whole of the right hand side and so the left hand side is also divisible by 3.

References: Chapter 1 Pages 3-4

Exercise 1A Q. 3

Proof by exhaustion

If there are a finite number of possibilities then proving by exhaustion involves testing the assertion in every case. E.g. proof that if a 2 digit number is divisible by 3 then the number obtained by reversing the digits is also divisible by 3.

This can be done by exhaustion as there are only a small number of such numbers (30 in all).

Proof that a number is divisible by 3 if the sum of the digits is divisible by 3 cannot be done by exhaustion.

Prove that there is only one prime number that is 1 less than a perfect square.

Consider any number n. Its square is n^2 and one less is $n^2 - 1 = (n - 1)(n + 1)$.
Thus $n^2 - 1$ is a product of two numbers and can therefore only be prime if one of those numbers is 1.
The only possibility is when $n = 2$ and $n - 1 = 1$.
So there is only one such number– when $n = 2$ and one less than its square is 3.
For all other values of n, one less than its square is a product of two numbers and is therefore not prime.

References: Chapter 1 Pages 4-5

Exercise 1A Q. 5

Proof by contradiction

"Either it is or it isn't".

If you can show that "it isn't" is not correct then by default "it is" must be right.

E.g. prove that $\sqrt{2}$ is irrational.

Assume that $\sqrt{2}$ is rational. I.e. that it can be expressed as a fraction $\frac{a}{b}$ where a and b are co-prime (that is, they have no common factors).

$\sqrt{2} = \frac{a}{b} \Rightarrow 2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2$

i.e. a^2 is even, which means that a is even.

So write $a = 2k \Rightarrow a^2 = 4k^2 = 2b^2$

$\Rightarrow b^2 = 2k^2$ i.e. b^2 is even, which means that b is even.

So both a and b are even which contradicts the assertion that a and b are co-prime.

So $\sqrt{2}$ cannot be written as a fraction and so is not rational.

References: Chapter 1 Pages 6-7

Exercise 1A Q. 8

Disproof by the use of a counter-example

For a theorem to be true it must be true in every case. To show that it is not in just one case is therefore sufficient to show that the theorem is not true.

E.g. Is $n^2 + n + 41$ prime for all positive n?

Substituting $n = 0, 1, 2$ and 3 gives 41, 43, 47, 53, all of which are prime.
It might therefore be assumed that the expression is prime for all n.
But when $n = 41$, $n^2 + n + 41 = 41^2 + 41 + 41$
$= 41 \times 43$ which is not prime.

References:
Chapter 2
Pages 8 - 11

Logarithms

From C2 the following laws for logarithms were derived.

$\log x + \log y = \log xy$

$\log x - \log y = \log \dfrac{x}{y}$

$\log x^n = n \log x$

$\log 1 = 0,\ \log_a a = 1$

$\log \dfrac{1}{x} = \log 1 - \log x = -\log x$

E.g. log2 + log3 = log6
Also ln2 + ln3 = ln6

E.g. log5 + log2 = log10 = 1
Also ln5 + ln2 = ln10 ≠ 1

E.g. log5 + 3log2 = log5 + $\log 2^3$ = log5 + log8 = log40

E.g. log3 + log9 = log3 + $\log 3^2$ = log3 + 2log3 = 3log3

E.g. log5 − 3log2 = (log10 − log2)− 3log2 = 1 − 4log2
= 1-log16 = log0.0625

E.g. make t the subject of the formula $T = T_0 + Ae^{-kt}$

$T - T_0 = Ae^{-kt} \Rightarrow \dfrac{T - T_0}{A} = \mathrm{e}^{-kt} \Rightarrow \mathrm{e}^{kt} = \dfrac{A}{T - T_0}$

$\Rightarrow kt = \ln \dfrac{A}{(T - T_0)} \Rightarrow t = \dfrac{1}{k} \ln \dfrac{A}{(T - T_0)}$

References:
Chapter.2
Pages 12-13

Natural Logarithms and the exponential Function

The number e is called the exponential number and the logarithm to the base e is known as the natural logarithm and is denoted ln$x$.

It can be seen that $\int \dfrac{1}{x}\mathrm{d}x = \log_e x$ where e = 2.718....

Then $\int_1^e \dfrac{1}{x}\mathrm{d}x = 1.$

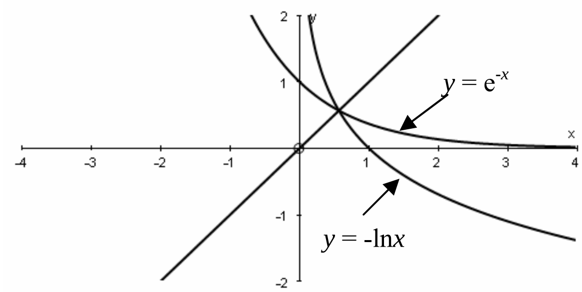
Exercise 2A
Q. 2

If y = ln$x$ then $x = \mathrm{e}^y$
So $y = \mathrm{e}^x$ is the inverse of y = ln$x$
(For the development of inverse functions see topic 3.)

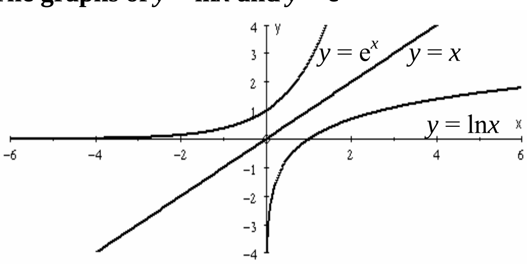
E.g. Sketch the graph of $y = \mathrm{e}^{-x}$ and its inverse.

$y = \mathrm{e}^{-x} = \dfrac{1}{\mathrm{e}^x} \Rightarrow \ln y = \ln\left(\dfrac{1}{\mathrm{e}^x}\right) = \ln 1 - \ln \mathrm{e}^x = 0 - x$

So the inverse is $y = -\ln x$



The graphs of y = ln$x$ and $y = \mathrm{e}^x$



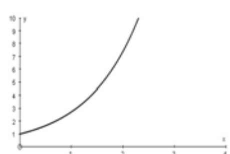
E.g. The population of rabbits on an island is modelled by the formula $P = 500 - 400\mathrm{e}^{-0.5t}$ where t is years after the start of counting.

(i) Sketch a graph of the function
(ii) How many rabbits are there initially?
(iii) Calculate the number of rabbits on the island after 5 years.
(iv) What is the long-term population according to this model?

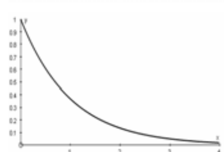
Exercise 2A
Q. 7, 10

Exponential growth and decay

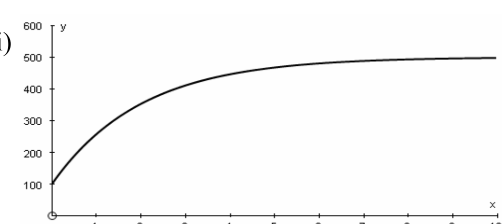
The graph of $y = \mathrm{e}^x$ is known as exponential growth.



The graph of $y = \mathrm{e}^{-x}$ is known as exponential decay.



(i)



(ii) When $t = 0$, $P = 500 - 400 = 100$
(iii) When $t = 5$, $P = 500 - 400\mathrm{e}^{-2.5} = 500 - 32.8 = 467$
(iv) When $t \Rightarrow \infty$, $P \Rightarrow 500 - 0 = 500$

References:
Chapter 3
Pages 19-23

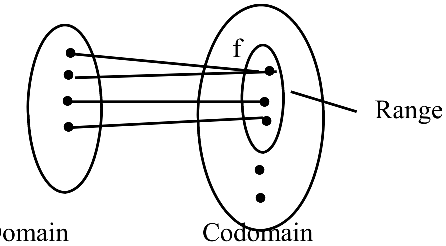
Exercise 3A
Q. 1(v), 3(i), 4(vi)

Terminology

A ***mapping*** is a rule which associates two sets of items.
The ***object*** maps onto the ***image***.
The set of possible objects forms the ***domain***.
The set of possible images forms the ***co-domain.***
The set of actual images is the ***range***.
The range is a sub-set of the co-domain.
A mapping can be ***one to one, many to one, one to many*** or ***many to many.***
If there is only one possible image for each object then the mapping is called a ***function***.

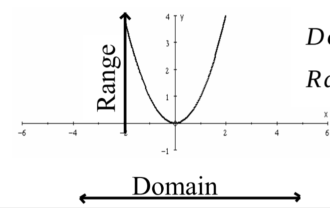
Notation for functions:

$y = x^3 \qquad f(x) = x^3 \qquad f : x \to x^3$



f is a many to one mapping and so is a function.

E.g. f: $x \to x^2$



$Domain : x \in \mathbb{R}$

$Range : y \in \mathbb{R}^+$

References:
Chapter 3
Pages 25-28

Exercise 3B
Q. 3, 4

References:
Chapter 3
Pages 30-32

Exercise 3C
Q. 1

Using transformations to sketch the curves of functions

$y = f(x-a)$ is the curve $y = f(x)$ translated a units in the +ve x direction.

$y = f(x) + a$ is the curve $y = f(x)$ translated a units in the +ve y direction.

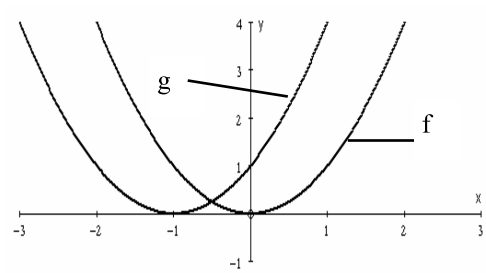
$y = af(x)$ for $a > 0$ is a one-way stretch of the curve $y = f(x)$ of scale factor a parallel to the y axis.

$y = f(ax)$ for $a > 0$ is a one-way stretch of the curve $y = f(x)$ of scale factor $\frac{1}{a}$ parallel to the x axis.

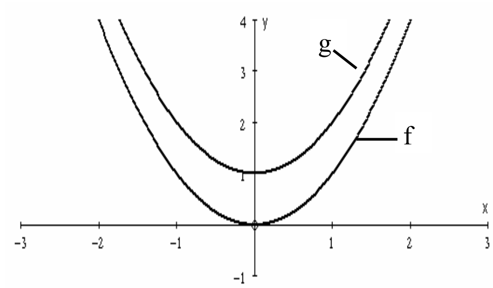
$y = f(-x)$ is a reflection of the curve $y = f(x)$ in the y axis.

$y = -f(x)$ is a reflection of the curve $y = f(x)$ in the x axis.

E.g. f:$x \to x^2$ g:$x \to (x + 1)^2$ i.e. g(x) = f(x + 1)



E.g. f:$x \to x^2$ g:$x \to x^2 + 1$ i.e. g(x) = f(x) + 1



References:
Chapter 3
Pages 32-33

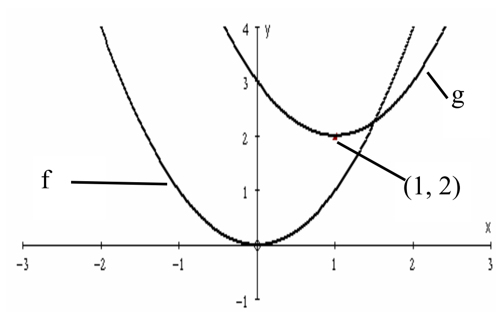
Exercise 3C
Q. 5

Quadratic functions

$y = x^2 + bx + c \equiv (x-t)^2 + s$

So $y = x^2 + bx + c$ is a translation of $y = x^2$ by t units in the +ve x direction and s units in the +ve y direction.
This process of rewriting the quadratic function is called completing the square.

E.g. g: $x \to x^2 - 2x + 3 \to y = (x - 1)^2 + 2$
f: $x \to x^2$ so g is a translation of f by 1 unit in the +ve x direction and 2 in the +ve y direction.



Competence statements f1, f2, f3, f4

References: Chapter 3 Pages 36-38

Exercise 3D Q. 1(v),(vi)

Composite functions

A composite function is a function of a function.

$\mathrm{fg}(x) = \mathrm{f}(z)$ where $z = \mathrm{g}(x)$

e.g. $\mathrm{f}(x) = x + 1,\ \mathrm{g}(x) = x^2 \Rightarrow \mathrm{fg}(x) = \mathrm{f}(x^2) = x^2 + 1$

N.B. $\mathrm{gf}(x) = g(x+1) = (x+1)^2$

i.e. $\mathrm{gf}(x) \neq \mathrm{fg}(x)$

E.g. $\mathrm{f}(x) = 3x$, $\mathrm{g}(x) = x^2 + 1$

Find (i) $\mathrm{fg}(x)$ and (ii) $\mathrm{gf}(x)$.

(i) $\mathrm{fg}(x) = \mathrm{f}(x^2 + 1) = 3(x^2 + 1)$

(ii) $\mathrm{gf}(x) = \mathrm{g}(3x) = (3x)^2 + 1$

References: Chapter 3 Pages 39-45

Exercise 3D Quest. 3, 5

Inverse functions

If the function f maps x onto y then the inverse mapping is y onto x.

This mapping can only be a function if x is uniquely defined for y. In other words the function f must be one to one.

If $y = \mathrm{f}(x)$ then the inverse function $y = \mathrm{f}^{-1}(x)$ is the reflection in the line $y = x$.

The criterion is that given x_1 maps onto y_1, y_1 is the image of ***only*** x_1.

E.g. $\mathrm{f}(x) = x + 2 \Rightarrow \mathrm{f}^{-1}(x) = x - 2$

E.g. $\mathrm{f}(x) = 2x - 1 \Rightarrow \mathrm{f}^{-1}(x) = \dfrac{x+1}{2}$

E.g. $\mathrm{f}(x) = x^2$ in $\mathbb{R}$ has no inverse as f(-2) = f(2) = 4.

but $\mathrm{f}(x) = x^2$ in $\mathbb{R}^+$ has the inverse $\mathrm{f}^{-1} = \sqrt{x}$

References: Chapter 3 Pages 45-46

Inverse trigonometrical functions

Trigonometrical functions are many to one and so have no inverse unless the domain is restricted.

E.g. $\cos 60^0 = 0.5$

$\cos^{-1} 0.5$ has infinitely many solutions (e.g. 60^0, 300^0, 420^0…)

But in the range $0 < x < 180^0$ $\cos^{-1} 0.5 = 60^0$

(This unique value is called the Principal Value.)

References: Chapter 3 Pages 49-53

Exercise 3E Q. 2, 5, 7

Even, Odd and Periodic functions

An ***even function*** has the y axis ($x = 0$) as the axis of symmetry. (i.e. $\mathrm{f}(x) = \mathrm{f}(-x)$.)

An ***odd function*** has rotational symmetry of order 2 about the origin. ($\mathrm{f}(x) = -\mathrm{f}(-x)$.)

A ***periodic function*** is one where $\mathrm{f}(x+k) = \mathrm{f}(x)$ where the minimum value of k is called the ***period***.

E.g. $y = \cos x$ is even.

$y = \sin x$ is odd.

Both $y = \cos x$ and $y = \sin x$ are periodic with $k = 360^0$.

References: Chapter 3 Pages 56-59

The modulus function

If $y = \mathrm{f}(x)$ takes negative values as well as positive values then the function $y = \mathrm{g}(x)$ which takes the positive numerical value is called the modulus function.

We write $y = |\mathrm{f}(x)|$.

E.g. $|x| < 5 \Leftrightarrow -5 < x < 5$

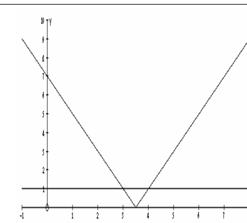
The graph of $y = |\mathrm{f}(x)|$ is obtained from $y = \mathrm{f}(x)$ by replacing values where $\mathrm{f}(x)$ is negative by equivalent positive values.

E.g. Solve $|2x - 7| < 1$

$|2x - 7| < 1 \Rightarrow -1 < 2x - 7 < 1$

$\Rightarrow 6 < 2x < 8 \Rightarrow 3 < x < 4$

This can be seen graphically.



E.g. $y = x + \dfrac{1}{x}$

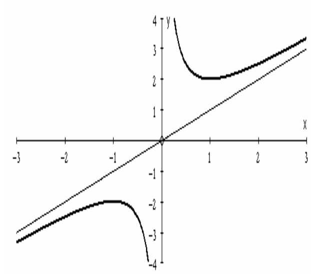
Does not cut either axis

It is an odd function

$x = 0$ is an asymptote,

as is $y = x$

As $x \Rightarrow \infty$, $y \Rightarrow x$



References: Chapter 3 Pages 60-61

Exercise 3F Q. 3

Curve Sketching

When sketching a curve the following processes should be adopted.

- Find where the curve crosses the axes
- Check for symmetry
- Find any asymptotes
- Examine the behaviour when $x \to \pm\infty$
- Look for any stationary points

References: Chapter 4 Pages 63-65

Exercise 4A Q. 1(i),(ix), 4

The Chain Rule

If $y = f(z)$ where $z = g(x)$ then $\frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$

E.g. If $y = (2x^3 + 3)^2$ then putting $z = 2x^3 + 3$ gives $y = z^2$

E.g. $y = \left(2x^2 + 3\right)^2$. Putting $z = 2x^2 + 3$ gives $y = z^2$

$\Rightarrow \frac{dy}{dx} = \frac{dy}{dz}.\frac{dz}{dx}$ and $\frac{dy}{dz} = 2z$ and $\frac{dz}{dx} = 4x$

$\Rightarrow \frac{dy}{dx} = 2\left(2x^2 + 3\right) \times 4x = 8x(2x^2 + 3)$

References: Chapter 4 Pages 65-66

Exercise 4A Q. 6

Rate of change

$y = f(x)$ where x and y both vary with time, t.

$\Rightarrow \frac{dy}{dt} = \frac{dy}{dx}.\frac{dx}{dt}$

E.g. A stone is dropped into a pond of still water. The ripples spread outwards in a circle at a rate of 10cm/sec. Find the rate of increase of area of ripples when r = 300cm.

$A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$\Rightarrow \frac{dA}{dt} = \frac{dA}{dr}.\frac{dr}{dt} = 2\pi r.10.$

When $r = 300$, $\frac{dA}{dt} = 6000\pi$

References: Chapter 4 Pages 68-70

Exercise 4B Q. 1(i),(ix), 3

The Product Rule

If $y = uv$ where $u = f(x)$ and $v = g(x)$

then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

e.g. If $y = x^2(3x^3 - 4x)$ then $u = x^2$ and $v = 3x^3 - 4x$

E.g. $y = \left(2x^2 + 3\right)\left(3x + 1\right)$

Put $u = \left(2x^2 + 3\right)$ and $v = \left(3x + 1\right)$

$\Rightarrow \frac{du}{dx} = 4x$ and $\frac{dv}{dx} = 3$

$\Rightarrow \frac{dy}{dx} = \left(2x^2 + 3\right)3 + \left(3x + 1\right)4x$

$= 18x^2 + 4x + 9$

References: Chapter 4 Pages 71-72

Exercise 4B Q. 1(iv), (vii), 4

The Quotient Rule

If $y = \frac{u}{v}$ where $u = f(x)$ and $v = g(x)$

then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

e.g. If $y = \frac{x^2}{(3x^3 - 4x)}$ then $u = x^2$ and $v = 3x^3 - 4x$

E.g.. $y = \frac{\left(2x^2 + 3\right)}{\left(3x + 1\right)}$

Put $u = \left(2x^2 + 3\right)$ and $v = \left(3x + 1\right)$

Then $\frac{du}{dx} = 4x$ and $\frac{dv}{dx} = 3$

$\Rightarrow \frac{dy}{dx} = \frac{\left(3x + 1\right)4x - \left(2x^2 + 3\right)3}{\left(3x + 1\right)^2} = \frac{6x^2 + 4x - 9}{\left(3x + 1\right)^2}$

References: Chapter 4 Pages 77-78

Exercise 4C Q. 3

Inverse functions

When $x = f(y)$ $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

Sometimes it is possible to make y the subject of the function, in which case $\frac{dy}{dx}$ can be found in the usual way.

This will usually not be possible, however, in examination questions.

E.g. $x = y^2 + 1 \Rightarrow \frac{dx}{dy} = 2y \Rightarrow \frac{dy}{dx} = \frac{1}{2y} = \frac{1}{2\sqrt{x - 1}}$

Note here:

$x = y^2 + 1 \Rightarrow y = \sqrt{x - 1} \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{x - 1}}$

E.g. $x = y^2 + y \Rightarrow \frac{dx}{dy} = 2y + 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y + 1}$

Note here that y can be made the subject of the function only with difficulty.

References:
Chapter 4
Pages 82-84

Exercise 4D
Q. 1(i),(iii), 2(i),(iii)

Natural logarithms and exponentials

$y = e^{ax} \Rightarrow \frac{dy}{dx} = ae^{ax}$

$y = \ln ax \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

$y = \ln x^n \Rightarrow \frac{dy}{dx} = \frac{n}{x}$

$y = \ln(f(x)) \Rightarrow \frac{dy}{dx} = \frac{f'(x)}{f(x)}$

E.g. $y = e^{4x} \Rightarrow \frac{dy}{dx} = 4e^{4x}$

$y = \ln(4x) \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

$y = \ln x^6 \Rightarrow \frac{dy}{dx} = \frac{6}{x}$

N.B. by rule of logs, $y = \ln(4x) = \ln 4 + \ln(x) \Rightarrow \frac{dy}{dx} = \frac{1}{x}$

References:
Chapter 4
Pages 91-94

Exercise 4E
Q. 1(i), 2(i), 3(i), 6

Differentiation of trig functions

f(x)	f '(x)	f(x)	f '(x)
$\sin x$	$\cos x$	$\sin ax$	$a\cos ax$
$\cos x$	$-\sin x$	$\cos ax$	$-a\sin ax$
$\tan x$	$\frac{1}{\cos^2 x}$	$\tan ax$	$\frac{a}{\cos^2 ax}$

x is measured in radians.

E.g. Differentiate the following:

(i) $y = \sin 2x + \cos 4x$

(ii) $y = x\sin 3x$ (using the Product rule)

(iii) $y = \frac{x}{\sin x}$ (using the Quotient rule)

(i) $\frac{dy}{dx} = 2\cos 2x - 4\sin 4x$

(ii) $\frac{dy}{dx} = \sin 3x + 3x\cos 3x$

(iii) $\frac{dy}{dx} = \frac{\sin x - x\cos x}{\sin^2 x}$

References:
Chapter 4
Pages 96-99

Exercise 4F
Q. 6

Implicit differentiation

This concerns functions where y is not the subject of the function

The function is differentiated using the chain rule.

$$\frac{d}{dx}g(y) = \frac{d(g(y))}{dy}.\frac{dy}{dx}$$

Note that when $y = f(x)$, $\frac{dy}{dx}$ will be a function of x

but when y is not the subject of the function, $\frac{dy}{dx}$ will be a function of x and y.

E.g. $y^2\sin x + y = 4$

$\Rightarrow \left(2y\frac{dy}{dx}\sin x + y^2\cos x\right) + \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{-y^2\cos x}{2y\sin x + 1}$

E.g. (See example on previous page)

$y = x - y^2 \Rightarrow \frac{dy}{dx} = 1 - 2y\frac{dy}{dx} \Rightarrow (1+2y)\frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = \frac{1}{(2y+1)}$

References:
Chapter 5
Pages 103-107

Integration by substitution
(Change of variable)

The integral is written in terms of a new variable *u:*

$$\int f(x)\,dx = \int g(u)\frac{du}{dx}\,dx$$

Indefinite integrals should be changed back after integrating to give an answer in terms of x.
Definite integrals should have the limits changed to correspond to the new variable.

Exercise 5A
Q. 1(i)(vii), 7

This method is most easily seen in two circumstances:
(i) When the "function of a function" is a linear function of x. e.g. $y = (2x-3)^4$
In this case you need to consider what number to multiply or divide by.

Exercise 5A
Q. 2(v)

(ii) When the function to be integrated looks like a product, but one part is the derivative of the other. E.g. $2x(x^2+3)^3$.

E.g. $I=\int(3x+4)^5\,dx$; Put $u=3x+4$; $\frac{du}{dx}=3$

$$\Rightarrow I=\int\frac{1}{3}u^5\,du=\frac{u^6}{18}=\frac{(3x+4)^6}{18}+c$$

E.g. $I=\int_0^1(2x+3)^2\,dx$; Put $u=2x+3$; $\frac{du}{dx}=2$

When $x=0,\ u=3$; when $x=1,\ u=5$

$$\Rightarrow I=\int_3^5 u^2\frac{1}{2}\,du=\left[\frac{u^3}{6}\right]_3^5=\frac{1}{6}\left(5^3-3^3\right)=\frac{49}{3}$$

Integration by inspection

If $I = \int f'(x)\left(f(x)\right)^n\,dx$, then $I=\frac{\left(f(x)\right)^{n+1}}{n+1}+c$

E.g. $I=\int 2x\sqrt{x^2+3}\,dx$

where $f(x)=x^2+3$ and $f'(x)=2x$.

This can be seen as the reverse of the Chain Rule.

E.g. $I=\int x^2\sqrt{x^3+3}\,dx = \frac{1}{3}\int 3x^2\sqrt{x^3+3}\,dx$

where $f(x)=x^3+3$ and $f'(x)=3x^2$.

$$\Rightarrow I=\frac{1}{3}.\frac{(x^3+3)^{\frac{3}{2}}}{\frac{3}{2}}+c=\frac{2}{9}(x^3+3)^{\frac{3}{2}}+c$$

E.g. $I = \int_1^2 x(x^2+1)^2\,dx = \frac{1}{2}\int_1^2 2x(x^2+1)^2\,dx$

where $f(x)=x^2+1$ and $f'(x)=2x$.

$$I=\frac{1}{2}\left[\frac{(x^2+1)^3}{3}\right]_1^2=\frac{1}{2}\left(\frac{5^3}{3}-\frac{2^3}{3}\right)=\frac{117}{6}$$

E.g. $I = \int 2x\sqrt{x^2+3}\,dx$

where $f(x) = x^2+3$ and $f'(x) = 2x$.

$$\Rightarrow I = \int f'(x)\left[f(x)\right]^{\frac{1}{2}}\,dx$$

$$= \frac{\left[f(x)\right]^{\frac{3}{2}}}{\frac{3}{2}}+c = \frac{2}{3}\left(x^2+3\right)^{\frac{3}{2}}+c$$

E.g. $I = \int x\left(3x^2+5\right)^4\,dx$

where $f(x) = 3x^2+5$ and $f'(x) = 6x$.

$$\Rightarrow I = \frac{1}{6}\int 6x.\left(3x^2+5\right)^4\,dx$$

$$= \frac{1}{6}\frac{\left(3x^2+5\right)^5}{5}+c = \frac{\left(3x^2+5\right)^5}{30}+c$$

E.g. $I = \int_0^1 x^2\left(x^3+1\right)^2\,dx$

where $f(x) = x^3+1$ and $f'(x) = 3x^2$.

$$\Rightarrow I = \frac{1}{3}\int_0^1 3x^2\left(x^3+1\right)^2\,dx$$

$$= \left[\frac{1}{3}\frac{\left(x^3+1\right)^3}{3}\right]_0^1 = \frac{1}{3}\left(\frac{2^3}{3}-\frac{1}{3}\right) = \frac{7}{9}$$

E.g. Differentiate $x\ln x$ and hence find $\int_1^2 \ln x\,dx$

$$\frac{d}{dx}(x\ln x)=\ln x + x.\frac{1}{x}=\ln x+1$$

$$\Rightarrow \int_1^2 \ln x\,dx=\int_1^2(\ln x+1-1)\,dx$$

$$=\int_1^2(\ln x+1)\,dx-\int_1^2 1\,dx$$

$$=\left[x\ln x - x\right]_1^2 = 2\ln 2-2-(\ln 1-1)=2\ln 2-1$$

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Competence statements c3, c4, c5, c6, c10

References:
Chapter 5
Pages 110-114

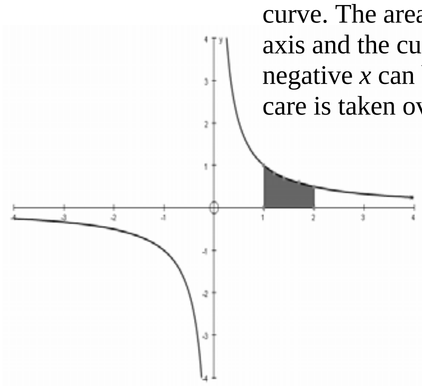
Exercise 5B
Q. 1(i),(v), 2(i),(v), 6

Integration involving exponentials and logarithms

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \frac{1}{x}\,dx = \left|\ln x\right| + c$$

Note that the integral represents the area under the curve. The area between the x axis and the curve $y = {}^{1}/_{x}$ for negative x can be found if care is taken over the signs.



E.g. $\int \frac{5}{x}\,dx = 5\ln x + c$

E.g. $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + c$

E.g. $\int_2^3 \frac{1}{2x+3}\,dx = \left[\frac{1}{2}\ln(2x+3)\right]_2^3$

$= \frac{1}{2}(\ln 9 - \ln 7) = \frac{1}{2}\ln\frac{9}{7}$

E.g. Find $\int_2^3 \frac{x^2+3}{x+3}\,dx$

$\frac{x^2+3}{x+3} = \frac{x^2+3x-3x+3}{x+3} = x-3+\frac{12}{x+3}$

$\Rightarrow \int_2^3 \frac{x^2+3}{x+3}\,dx = \int_2^3 \left(x-3+\frac{12}{x+3}\right)dx$

$= \left[\frac{x^2}{2} - 3x + 12\ln(x+3)\right]_2^3 = \left(\frac{9}{2} - 9 + 12\ln 6\right) - (2 - 6 + 12\ln 5)$

$= 12\ln\frac{6}{5} - \frac{1}{2}$

References:
Chapter 5
Pages 123-124

Exercise 5C
Q. 2(i), 3(i), 4(i)

Integration of trig functions

$$\int \sin x\,dx = -\cos x + c \qquad \int \sin ax\,dx = -\frac{1}{a}\cos ax + c$$

$$\int \cos x\,dx = \sin x + c \qquad \int \cos ax\,dx = \frac{1}{a}\sin ax + c$$

E.g. $\int(\sin 2x + \cos 4x)\,dx$

$= -\frac{1}{2}\cos 2x + \frac{1}{4}\sin 4x + c$

E.g. $\int_0^{\pi/6} \cos 3x\,dx = \left[\frac{1}{3}\sin 3x\right]_0^{\pi/6} = \frac{1}{3} - 0 = \frac{1}{3}$

References:
Chapter 5
Pages 125-130

Exercise 5D
Q. 2(i),(iii), 4

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

This formula is used to integrate products.

e.g. $\int x\sin x\,dx$, $\int xe^x\,dx$

E.g. $I = \int x\cos x\,dx$; $\quad u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

$\Rightarrow I = x\sin x - \int 1.\sin x\,dx = x\sin x + \cos x + c$

References:
Chapter 5
Pages 131-132

Exercise 5E
Q. 1(i),(iv), 3

Definite integration by parts

$$\int_a^b u\frac{dv}{dx}\,dx = \left[uv\right]_a^b - \int_a^b v\frac{du}{dx}\,dx$$

E.g. $\int_0^2 xe^{2x}\,dx = \left[\frac{1}{2}xe^{2x}\right]_0^2 - \frac{1}{2}\int_0^2 e^{2x}\,dx$

$\{u = x \Rightarrow \frac{du}{dx} = 1,\ \frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}\}$

$= (e^4 - 0) - \frac{1}{4}\left[e^{2x}\right]_0^2$

$= e^4 - \frac{1}{4}(e^4 - 1) = \frac{3}{4}e^4 + \frac{1}{4}$

Competence statements c3, c4, c5, c6, c10

Note that this topic is the subject of the component of coursework attached to this module. There will be no questions on this topic in the examination.

References:
Chapter 6
Pages 135-154

Misconceptions and common errors seen in C3 coursework

Terminology

Students commonly refer to an expression (e.g. $x^3 + x - 7$) or a function (e.g. $y = x^3 + x - 7$) as an equation. What you are doing is to solve the equation $f(x) = 0$ and to illustrate it you are going to draw the graph of $y = f(x)$. Take care to use the correct words throughout your coursework!

Numerical solutions

There are some equations that can be solved analytically and some which cannot. Where an analytical solution is known to exist it should be employed to solve the equation. When an equation cannot be solved analytically a numeric method may be employed. The one is not inferior to the other- they are used in different circumstances.

Exercise 6A
Q. 2(i),(iii), 4

Error bounds

A numeric solution without error bounds is useless. Many students will work a numeric method to produce a root such as $x = 1.234561$ and then assert that it is correct to 3 decimal places or even not give any error bounds.

Error bounds are often stated and "justified" by scrutiny of the digits within consecutive iterates. The decimal search methods and iterative methods which, when illustrated graphically, display a cobweb diagram, have inbuilt error bounds (but even then need to be stated appropriately!) but an iterative method which is illustrated by a staircase diagram does not, and in this case error bounds need to be established by a change of sign. This is true also for a root found by the Newton-Raphson method.

Exercise 6B
Q. 1, 4

Failure of methods

In each case you are asked to demonstrate a failure.

The Newton-Raphson method requires as a condition for use "that the initial value for x be close to the root". If, therefore, the value of $x_0 = 1$ yields a root in the range [1,2] then it would not be appropriate to suggest that the method has failed if a starting value of $x_0 = 3$, say, does not yield the same root.

Exercise 6C
Q. 2, 5

The importance of graphical illustrations

It is crucial that you connect the graphical understanding to what is being done numerically.

The problems about the misuse of terminology described above may be overcome with a clear understanding of the connection between the two. Indeed, if graphical solutions are introduced properly as a method of solving equations then much of the difficulty will be overcome.

For instance, to solve $x^3 + x - 7 = 0$, draw the graph of the function $y = x^3 + x - 7$. Where this graph crosses the x-axis is the point where $y = 0$ and so gives an approximate value for the root of the equation.

The employment of such a process can also act as a check to the work being done. Often very able students make arithmetic (or algebraic) errors which they do not pick up because of their inability to see visually what is going on.