CORE 4 Summary Notes

1 Rational Expressions 1

- Factorise all expressions where possible
- Cancel any factors common to the numerator and denominator

$$\frac{x^2 + 5x}{x^2 - 25} = \frac{x\cancel{(x+5)}}{\cancel{(x+5)}(x-5)} = \frac{x}{x-5}$$

- To add or subtract - the fractions must have a **common denominator**

$$\frac{6}{x} + \frac{3}{x(x-4)} = \frac{6(x-4)}{x(x-4)} + \frac{3}{x(x-4)} = \frac{6x - 24 + 3}{x(x-4)} = \frac{6x-21}{x(x-4)}$$

- To divide by a rational expression you can multiply by it's reciprocal.

2 Rational Expressions

- REMAINDER THEOREM

  When p(x) is divided by (ax-b) the remainder is $p(\frac{b}{a})$

  Note that $x = \frac{b}{a}$ is the solution of the equation $ax - b = 0$

E.g Show that 2x –1 is a factor of $2x^3 + 7x^2 - 14x + 5$

$2x - 1 = 0$

$x = 0.5$

$$2(0{\cdot}5)^3 + 7(0{\cdot}5)^2 - 14(0{\cdot}5) + 5 = 0$$

Must be a factor as the remainder is zero

- An Algebraic fraction is 'PROPER' is the degree of the polynomial that is the numerator is less than the degree of the polynomial that is the denominator.

E.g.1 $$\frac{x+6}{x+2} = \frac{x+2+4}{x+2} = \frac{x+2}{x+2} + \frac{4}{x+2} = 1 + \frac{4}{x+2}$$

2 $$\frac{x^2 + 6x - 3}{x+2} = \frac{(x+2)(x+4) - 11}{x+2} = x + 4 - \frac{11}{x+2}$$

3 $$\frac{x^3 - 2x^2 + x + 5}{x^2 - 3} = \frac{(x^2-3)(x-2) + 4x - 1}{x^2 - 3} = x - 2 + \frac{4x-1}{x^2-3}$$

- An **identity** is a statement that is true for all values of x for which the statement is defined

- PARTIAL FRACTIONS
  Any proper algebraic fraction with a denominator that is a product of distinct linear factors can be written as partial fractions as the sum of proper fractions whose denominators are linear factors.

$\frac{5x + 1}{(x - 1)(2x + 1)(x - 5)}$ can be expressed in the form $\frac{A}{x - 1} + \frac{B}{2x + 1} + \frac{C}{x - 5}$

E.g $\frac{5}{(x - 2)(x + 3)} = \frac{A}{x - 2} + \frac{B}{x + 3} = \frac{A(x + 3) + B(x - 2)}{(x - 2)(x + 3)}$

Looking at the numerators A(x + 3) + B(x - 2) = 5

x= 2 5A = 5 so A = 1

x= -3 -5B = 5 so B = -1

$\frac{5x + 1}{(x - 1)(2x + 1)^2}$ can be expressed in the form $\frac{A}{x - 1} + \frac{B}{2x + 1} + \frac{C}{(2x + 1)^2}$

REPEATED FACTOR

3 Parametric Equations

- Two equations that separately define the x- and y- coordinates of a graph in terms of a third variable.
- The third variable is called the **parameter**

  $x = t + 4 \quad y = 1 - t^2$

- To convert a pair of parametric equations to single Cartesian equation, eliminate the parameter.

E.g. $x = t + 4 \rightarrow t = x - 4$ so $t^2 = x^2 - 8x + 16$

$y = 1 - t^2$

$y = 1 - (x^2 - 8x + 16)$

$y = 8x - x^2 - 15$

- CIRCLE and ELLIPSE
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The curve $x = r\cos\theta \quad y = r\sin\theta$ is a circle with radius r and centre the origin

The curve $x = r\cos\theta + p \quad y = r\sin\theta + p$ is a circle with radius r and centre (p,q)

The curve $x = a\cos\theta \quad y = b\sin\theta$ is an ellipse, centre the origin.
Its width is 2a and its height is 2b units.

Its Cartesian equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

e.g. A curve is given by $x = 2\cos\theta + 3 \quad y = \sin\theta - 1$
Find the Cartesian equation of the curve

$$\cos\theta = \frac{x-3}{2} \qquad \sin\theta = y + 1$$

Using the identity $\sin^2\theta + \cos^2\theta = 1$

$$(y+1)^2 + \left(\frac{x-3}{2}\right)^2 = 1$$

- TRANSFORMING PARAMETRIC GRAPHS

Translation $\begin{bmatrix} a \\ b \end{bmatrix}$ add a to the x function and b to the y function

Stretch
x direction, multiply the x function by the required factor
y direction, multiply the y function by the required factor

Reflection
y- axis, multiply the x function by –1
x- axis, multiply the y function by –1

4 The Binomial Theorem

The Binomial Expansion

$$(1 + ax)^n = 1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \frac{n(n-1)(n-2)}{3!}(ax)^3 + \ldots\ldots$$

is valid for negative and fractional values of n for **$|ax|<1$**

E.g. Expand $\dfrac{1+3x}{(1+x)^3}$ $|x| < 1$ in ascending powers of x as far as x^3

$$\frac{1+3x}{(1+x)^3} = (1+3x)(1+x)^{-3}$$

$$= (1+3x)\left(1 + (-3)x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3\right)$$

$$(1+3x)(1 - 3x + 6x^2 - 10x^3 \ldots)$$

$$= 1 - 3x^2 + 8x^3$$

- For an expansion of an expression such as $\frac{x + 5}{(3 - x)(1 + 3x)}$ split the expression into PARTIAL FRACTIONS before attempting an expansion.

5 Trigonometric Formulae

ADDITION FORMULAE

sin (A+B) = sin A cos B + sin B cos A

sin (A-B) = sin A cos B – sin B cos A

cos (A+B) = cos A cos B – sin A sin B

cos (A-B) = cos A cos B + sin A sin B

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

DOUBLE ANGLE FORMULAE

sin 2A = 2sin A cos A

$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

- asin x + bcos x can be written in the form

$r\sin(x + \alpha)$ where $a = r\cos\alpha$ and $b = r\sin\alpha$

$r\cos(x - \alpha)$ where $a = r\sin\alpha$ and $b = r\cos\alpha$

$r^2 = a^2 + b^2$

e.g Find the maximum value of the expression 2 sin x + 3 cos x by expressing it in the form $r\sin(x + \alpha)$

$r^2 = a^2 + b^2$

$r^2 = 2^2 + 3^2 = 13$

$r = \sqrt{13}$

$\frac{r\sin\alpha}{r\cos\alpha} = \frac{3}{2}$

$\tan\alpha = \frac{3}{2}$

$\alpha = 56$

$2\sin x + 3\cos x = \sqrt{13}\sin(x + 56)$

Maximum value is $\sqrt{13}$ which occurs when sin(x+56) =1 ➔ x = 34

- asinx – bcos x can be written in the form

$r\sin(x - \alpha)$, where $a = r\cos\alpha$ and $b = r\sin\alpha$

$r\cos(x + \alpha)$, where $a = -r\sin\alpha$ and $b = -r\cos\alpha$

$r^2 = a^2 + b^2$

- Both of the above are useful in SOLVING EQUATIONS.

6 Differential Equations

- Key points from core 3

The derivative **of e^{ax} is ae^{ax}**

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$$

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + c$$

- An equation that involves a derivative is called a **Differential Equation**. They are used to model problems involving rates of change.

  e.g The rate of growth of a population is proportional to the size of the population.
  Let the population at time t to be P

$$\frac{dP}{dt} = kP \text{ where k is a constant}$$

- SEPARATING THE VARIABLES – a method of solving differential equations.

  Find the general solution of $\frac{dy}{dx} = 2x(y+4) \quad y > 0$

  1) Separate the variables

$$\frac{dy}{y+4} = 2x\,dx$$

  2) Integrate both sides

$$\int \frac{1}{y+4}\,dy = \int 2x\,dx$$

$$\ln(y+4) = x^2 + c$$

$$y + 4 = e^{x^2+c} = e^{x^2}e^c = Ae^{x^2} \quad \text{where } A = e^c$$

$$y = Ae^{x^2} - 4$$

- EXPONENTAL GROWTH
  An equation of the form $y = ae^{bt}$ $(a > 0 \quad b > 0)$ represents exponential growth

- EXPONENTAL DECAY
  An equation of the form $y = ae^{-bt}$ $(a > 0 \quad b > 0)$ represents exponential growth

- The equation

  $y = c \pm ae^{-bt}$ $(a > 0 \quad b > 0)$ represents a process in which the value of y gets closer to c as $t \Rightarrow \infty$
- The expression a^x is equivalent to $e^{(\ln a)x}$.
  The derivative of a^x is $(\ln a)a^x$.

7 Differentiation

- PARAMETRIC EQUATIONS

$$\frac{dy}{dx} = \left(\frac{dy}{dt}\right) \div \left(\frac{dx}{dt}\right)$$

e.g. $x = 2t \qquad y = t^3$

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 3t^2$$

$$\frac{dy}{dx} = \frac{3t^2}{2}$$

- FUNCTIONS DEFINED IMPLICITLY,
  To find the gradient of a graph defined implicitly we need to be able to differentiate, with respect to x, expressions containing both x and y.

$$\frac{d}{dx}(\quad) = \frac{d}{dy}(\quad)\frac{dy}{dx}$$

e.g.

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx}$$

$$= 2y\frac{dy}{dx}$$

If a function includes a product involving x and y then the 'Product Rule' is needed.

$$\frac{d}{dx}(xy^2) = x\frac{d}{dx}(y^2) + y^2\frac{d}{dx}(x)$$

$$= \ = 2xy\frac{dy}{dx} + y^2$$

8 Integrals

- USING PARTIAL FRACTIONS
  Functions such as $\frac{x^2}{(x-2)(x+3)}$, $\frac{x+5}{(2x+1)(x-1)^2}$
  can be integrated using partial fractions.

  If the degree of the numerator $\geq$ degree of denominator, then there will be a quotient + partial fractions

$$\int \frac{x+9}{(x-3)(x+1)}dx = \int \frac{3}{x-3}dx - \int \frac{2}{x+1}dx$$

$$= 3\int \frac{1}{x-3}dx - 2\int \frac{1}{x+1}dx$$

$$= 3\ln|x-3| - 2\ln|x+1| + c$$

- USING TRIGONOMETRICAL IDENTITIES

$$\int \sin x \cos x \, dx = \frac{1}{2}\int \sin 2x$$

Using the identity
$2\sin x \cos x = \sin 2x$

$$= -\frac{1}{4}\cos 2x + c$$

$$\int \cos^2 x \, dx = \frac{1}{2}\int (\cos 2x + 1)\, dx$$

Using the identity
$\cos 2x = 2\cos^2 x - 1$

$$\int \sin^2 \frac{x}{2} \, dx = \frac{1}{2}\int (1 - \cos x)$$

Replacing x by ½ x in the identity
$\cos 2x = 1 - 2\sin^2 x$

9 Vectors

- A vector has two properties :
  Magnitude (or size)
  and *Direction*

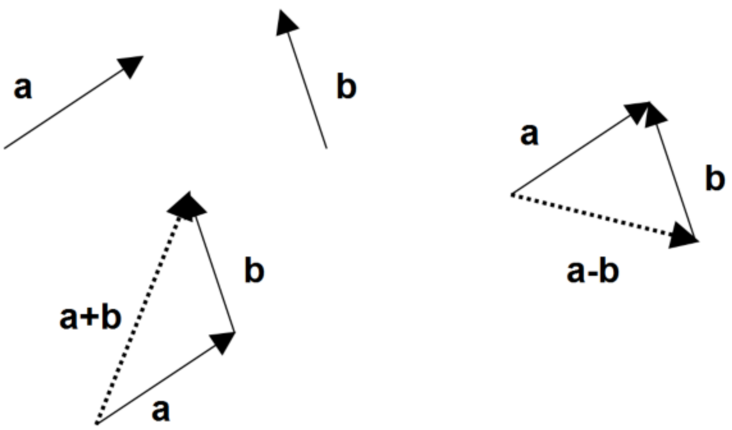
- Vectors with the same magnitude and direction are equal.

- The modulus of a vector is its magnitude.
  The modulus of the vector **a** is written **|a|**

- Any vector parallel to the vector **a** may be written as λ**a** where λ is a non-zero real number and is sometimes called a scalar multiple of **a.**

  \- **a** has the same magnitude but is in the opposite direction to **a.**

- Vectors can be added and subtracted using the 'triangle law'.



- Vectors can be written in column vector form such as $\begin{pmatrix} 3 \\ -7 \\ 1 \end{pmatrix}$

- A *unit vector* is a vector with a magnitude of 1.

The vectors **i, j** and **k** are unit vectors in the direction of the x-, y- and z- axes respectively.

As column vectors **i =** $\begin{pmatrix}1\\0\\0\end{pmatrix}$ **, j =** $\begin{pmatrix}0\\1\\0\end{pmatrix}$ **, k =** $\begin{pmatrix}0\\0\\1\end{pmatrix}$

Vectors can be written as linear combinations of these unit vectors,

e.g. $\begin{pmatrix}3\\-7\\1\end{pmatrix}$ = 3**i** – 7**j** + **k**

The *magnitude (or modulus)* of the vector $\begin{pmatrix}3\\-7\\1\end{pmatrix}$ = 3**i** – 7**j** + **k** is

$$\sqrt{3^2 + (-7)^2 + 1^2} = \sqrt{59}$$

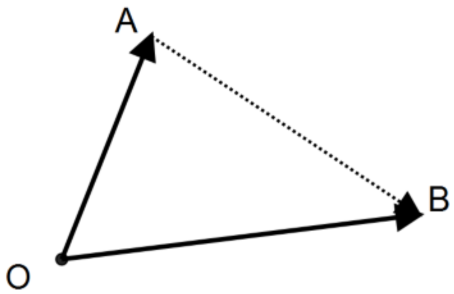
The *distance* between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- For every point P there is a unique vector $\overrightarrow{OP}$ (where O is a fixed origin) which is called the *position vector* of the point P.

  The point with coordinates (x, y, z) has position vector $\begin{pmatrix}x\\y\\z\end{pmatrix}$

  For two points A and B with position vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$ the vector AB is given by



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

- The general form of a *vector equation of a line* is

$$\mathbf{r} = \mathbf{p} + \lambda\mathbf{d}$$

r is the *position vector* of any point on the line,
p is the *position vector* of a particular point on the line,
λ is a scalar parameter,
d is any vector parallel to the line (called a **direction vector**)

N.B. Since **p** and **d** are not unique then your equation might not look identical to the one given in the back of the book!

- Lines are parallel if their direction vectors are parallel.

- In 3 – dimensions, a pair of lines may be parallel,
  or they intersect,
  or they are skew.

- To show that two straight lines intersect, find values of the parameters in their vector equations that produce the same point on each line. If no such values can be found then the lines are skew.

- The angle θ between two vectors is defined as the one formed when the vectors are placed 'tail to tail' or 'head to head' so that $0 \le \theta \le 180^0$

- **<u>Scalar (or dot) product</u>**

  For two vectors **a** and **b**

  <u>Definition 1</u> **a.b =** $a_1 b_1 + a_2 b_2 + a_3 b_3$

  <u>Definition 2</u> **a.b** = |**a**| |**b**| cos θ (where θ is the angle between the vectors)

  Combining the two definitions gives the angle between two vectors:

  $$\cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|\mathbf{a}|\,|\mathbf{b}|}$$

- Vectors **a** and **b** are perpendicular if and only if **a.b =** 0

One angle between two straight lines is the angle between their direction vectors.