C4 Summary of Notes

Partial Fractions

- $$\frac{ax+b}{(cx+d)(ex+f)} \equiv \frac{A}{(cx+d)} + \frac{B}{(ex+f)}$$
- $$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} \equiv \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

improper fractions:

- $$\frac{\text{quadratic}}{(ax+b)(cx+d)} = A + \frac{B}{(ax+b)} + \frac{C}{(cx+d)}$$
- $$\frac{\text{cubic}}{(ax+b)(cx+d)} \equiv Ax+B + \frac{C}{(ax+b)} + \frac{D}{(cx+d)} \quad \text{etc.}$$

Binomial Theorem

for general index n, $(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

provided $|x| < 1 \Rightarrow -1 < x < 1$

and $(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n = a^n\left[1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \dots\right]$

provided $\left|\frac{x}{a}\right| < 1 \Rightarrow -a < x < a$

Differentiation

implicit: $\frac{d}{dx} f(y) = \frac{d}{dy} f(y) \cdot \frac{dy}{dx}$

exponential: $\frac{d}{dx} a^x = a^x \ln a$

general logs: $\frac{d}{dx} \log_a x = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) = \frac{1}{x \ln a}$

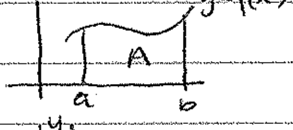
Connected rates of change: $\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}$

Parametric: for $x = f(t)$ and $y = g(t)$

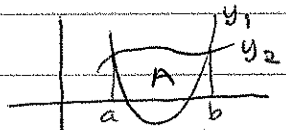
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

Integration

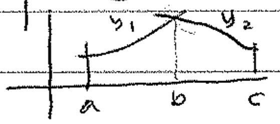
- areas



$$A = \int_a^b y\,dx$$

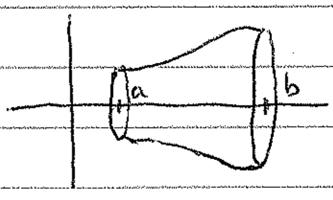
- 

$$A = \int_a^b (y_2 - y_1)\,dx$$

- 

$$A = \int_a^b y_1\,dx + \int_b^c y_2\,dx$$

volumes:



$$V = \pi \int_a^b y^2\,dx$$

Integration [x in radians]

Standard results:

y	$\int y\,dx$	$+c$ in each case
x^n	$\frac{x^{n+1}}{n+1}$	$n \neq -1$
$\frac{1}{x}$	$\ln\lvert x\rvert$	
e^x	e^x	
$\sin x$	$-\cos x$	
$\cos x$	$\sin x$	
$\sec^2 x$	$\tan x$	
$\operatorname{cosec}^2 x$	$-\cot x$	
$\operatorname{cosec} x \cot x$	$-\operatorname{cosec} x$	
$\sec x \tan x$	$\sec x$	
$\tan x$	$\ln\lvert\sec x\rvert$	
$\cot x$	$\ln\lvert\sin x\rvert$	
$\sec x$	$\ln\lvert\sec x + \tan x\rvert$	
$\operatorname{cosec} x$	$-\ln\lvert\operatorname{cosec} x + \cot x\rvert$	

Integration by Recognition $\lambda\int f'(x)\,dx = \lambda f(x) + c$

reversing the Chain Rule: $\lambda\int f'(g(x))g'(x)\,dx = \lambda f(g(x)) + c$

special cases:

(1) the "$\frac{1}{a}$" rule $\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b) + c$

in particular:

$$\int e^{kx}\,dx = \frac{1}{k}e^{kx} + c$$
$$\int \sin kx\,dx = -\frac{1}{k}\cos kx + c \quad \text{(etc)}$$
$$\int (ax+b)^n\,dx = \frac{1}{n+1}\cdot\frac{1}{a}(ax+b)^{n+1} + c \quad (n \neq -1)$$
$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln\lvert ax+b\rvert + c$$

(2)

- $\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$
- $\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$
- $\int f'(x)\sin(f(x))\,dx = -\cos(f(x)) + c$ (etc)
- $\int \sin x \cos^n x\,dx = -\frac{1}{n+1}\cos^{n+1}x + c$
- $\int \cos x \sin^n x\,dx = \frac{1}{n+1}\sin^{n+1}x + c$
- $\int f'(x)[f(x)]^n\,dx = \frac{1}{n+1}[f(x)]^{n+1} + c$

$\Big\}\ (n \neq -1)$

Integration by Parts

$$\int uv'\,dx = uv - \int u'v\,dx$$

How to choose:
- $\int x^n \times$ exponential — $u = x^n$, $v' = \exp$
- $\int x^n \times$ trig — $u = x^n$, $v' =$ trig

but
- $\int x^n \times \ln x$ $(n \neq -1)$ — $u = \ln x$, $v' = x^n$

Integration by Substitution

(1) Algebraic:
- if substitution not given, choose $u =$ a bracketed term
- differentiate the substitution to obtain dx in terms of du
- replace all x terms with their equivalents in terms of u
- change limits from x values to corresponding u-values.
- simplify and integrate.

(2) Trigonometric:
- as above, except:-
- use Pythagorean identity to simplify, eg to dissolve square root
- angles must be principal values in radians

Using Trig Identities

POWERS:
- even powers of sin, cos:

$$\int \sin^2 x\,dx = \tfrac{1}{2}\int (1-\cos 2x)\,dx = \tfrac{1}{2}\left[x - \tfrac{1}{2}\sin 2x\right] + c$$

$$\int \cos^2 x\,dx = \tfrac{1}{2}\int (1+\cos 2x)\,dx = \tfrac{1}{2}\left[x + \tfrac{1}{2}\sin 2x\right] + c$$

- odd powers of all trig, and all other cases:

use appropriate Pythagorean Identity and Integration by Recognition eg

$$\int \sin^3 x\,dx = \int \sin x(1-\cos^2 x)\,dx = \int \sin x - \sin x\cos^2 x\,dx = -\cos x + \tfrac{1}{3}\cos^3 x + c$$ etc.

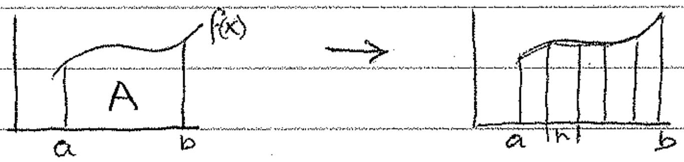
PRODUCTS: eg $\int \sin nx \sin mx \, dx$, $\int \sin nx \cos mx \, dx$ etc
use C3 addition formulae to express as a sum or difference first

Parametric Integration for $x = f(t)$ $y = g(t)$
use $dx = \frac{dx}{dt} dt$ and follow same procedure for integration by substitution

Separable Variable Differential Equations

given $\frac{dy}{dx} = f(x) \cdot g(y)$ the general solution is given by
$$\int \frac{1}{g(y)} dy = \int f(x) \, dx + c$$
the particular solution for which $x = x_0$ when $y = y_0$ is given by
$$\int_{y_0}^{y} \frac{1}{g(y)} dy = \int_{x_0}^{x} f(x) \, dx$$

Trapezium Rule



- n strips $\Rightarrow$ $n+1$ ordinates (y-values). $h = \frac{b-a}{n}$ = step length
- $A \simeq \frac{h}{2}$[first + last + 2(all the other y-values)]
- % error $= \frac{|\text{true} - \text{estimate}|}{\text{true}} \times 100\%$

Vectors
- $\vec{AB} = \vec{OB} - \vec{OA}$
- $\underline{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow |\underline{v}| = \sqrt{a^2 + b^2 + c^2}$
- unit vector $\hat{\underline{v}} = \frac{\underline{v}}{|\underline{v}|}$ so that $|\hat{\underline{v}}| = 1$

equation of line thru $\underline{a}$ in direction $\underline{b}$ is $\underline{r} = \underline{a} + \lambda \underline{b}$

Scalar product:
- $\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos\theta$
- $\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \Rightarrow \underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

$\Rightarrow \cos\theta = \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$. $\underline{a} \cdot \underline{b} = 0 \Leftrightarrow \underline{a} \perp \underline{b}$