MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

C4, Applications of Advanced Mathematics

(Version B—reference to new book)

Topic 1: Algebra

Topic 2: Trigonometry

Topic 3: Parametric Equations

Topic 4: Integration

Topic 5: Vectors

Topic 6: Differential Equations

(There is no reference to the Comprehension Task in this resource.)

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References:
Chapter 7
Pages 156-164

The Binomial Theorem

The general form for n a positive integer (from C1) is

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \dots\dots + b^n$$

The general form when n is not a positive integer is

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots\dots$$

provided $|x| < 1$

If the first term is not 1, then a factor must be taken out.
E.g.

$$(2+x)^n = 2^n\left(1+\frac{x}{2}\right)^n$$

which may be expanded by the Binomial Theorem provided $\left|\frac{x}{2}\right| < 1$.

Exercise 7A
Q. 1(i),(iii), 3

E.g. Expand $\frac{1}{\sqrt{1-2x}}$ as far as the term in x^3, stating the range of values of x for which the expansion is valid.

$$\frac{1}{\sqrt{1-2x}} = (1-2x)^{-\frac{1}{2}}$$

$$= 1 + \left(-\frac{1}{2}\right)(-2x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}(-2x)^2 + \frac{\left(-\frac{1}{2}\right)\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}(-2x)^3 + ..$$

$$= 1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \dots\dots \quad \text{provided } |-2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

E.g. Find a quadratic approximation for $\frac{x+1}{(x-2)^2}$.
stating the range of values of x for which the expression is valid.

$$\frac{x+1}{(x-2)^2} = (x+1)(x-2)^{-2} = (1+x)(2-x)^{-2} = (1+x)2^{-2}\left(1-\frac{x}{2}\right)^{-2}$$

$$= \frac{1}{4}(1+x)\left(1+(-2)\left(\frac{-x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{-x}{2}\right)^2+\dots..\right)$$

$$= \frac{1}{4}(1+x)\left(1+x+\frac{3}{4}x^2\right) = \frac{1}{4}\left(1+x+\frac{3}{4}x^2+x+x^2\right)$$

$$= \frac{1}{4}+\frac{1}{2}x+\frac{7}{16}x^2 \quad \text{provided that } |x| < 2$$

References:
Chapter 7
Pages 166-168
Pages 169-171

Rational Expressions

To add or subtract rational expressions

- Find the lowest common multiple of the denominators
- Make equivalent fractions and simplify the numerator

To solve equations involving fractions

- Clear fractions by multiplying each side by the LCM of the denominators.

Exercise 7B
Q. 1, 5, 11, 21

Exercise 7C
Q. 1(i),(iii), 5

E.g. Simplify $\frac{x}{x^2-4} - \frac{3}{x+2}$

$$\frac{x}{x^2-4} - \frac{3}{x+2} = \frac{x}{(x-2)(x+2)} - \frac{3}{x+2}$$

$$= \frac{x-3(x-2)}{(x-2)(x+2)} = \frac{2(3-x)}{(x-2)(x+2)}$$

E.g. Solve $\frac{x+1}{x-1} - \frac{x}{x+1} = \frac{7}{3}$

$$\times 3(x-1)(x+1) \Rightarrow 3(x+1)^2 - 3x(x-1) = 7(x-1)(x+1)$$

$$\Rightarrow 3x^2+6x+3-3x^2+3x = 7x^2-7$$

$$\Rightarrow 7x^2-9x-10 = 0 \Rightarrow (7x+5)(x-2) = 0$$

$$\Rightarrow x = 2 \text{ or } \frac{-5}{7}$$

References:
Chapter 7
Pages 173-180

Partial Fractions

Putting a rational function into partial fractions is the process of expressing it as the sum of fractions.
This process is only valid for proper fractions.

Type 1. Linear factors in the denominator.

e.g. $\frac{2}{(x+1)(x+2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+2)}$

Type 2. A quadratic factor in the denominator.

e.g. $\frac{2x+1}{(x^2+1)(x+2)} \equiv \frac{Ax+B}{(x^2+1)} + \frac{C}{(x+2)}$

Type 3. Repeated factors in the denominator.

e.g. $\frac{2x+1}{(x+1)^2(x+2)} \equiv \frac{A}{(x+1)} + \frac{B}{(x+1)^2} + \frac{C}{(x+2)}$

Exercise 7D
Q. 1(viii)

Exercise 7E
Q. 1(viii)

Exercise 7F
Q. 1(iii), 3

E.g. Split $\frac{7x+6}{(x+1)(x+2)}$ into partial fractions.

$$\frac{7x+6}{(x+1)(x+2)} \equiv \frac{A}{x+1} + \frac{B}{x+2}$$

where $A(x+2) + B(x+1) \equiv 7x+6$

$\Rightarrow A+B = 7$ and $2A+B = 6 \Rightarrow A = -1$ and $B = 8$

$$\Rightarrow \frac{7x+6}{(x+1)(x+2)} \equiv \frac{8}{x+2} - \frac{1}{x+1}$$

References: Chapter 8 Pages 183-186

Exercise 8A Q. 1(ii), 2(ii), 5

Reciprocal Functions

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta},\quad \sec\theta = \frac{1}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$
and $\tan^2\theta + 1 = \sec^2\theta$

E.g. $\sec 235^0 = \frac{1}{\cos 235^0} = \frac{1}{-\cos 45^0} = -\sqrt{2}$

E.g. Solve $2\sec^2 x + \tan x - 3 = 0$ in the range $0^0 \le x \le 360^0$

$2\sec^2 x + \tan x - 3 = 0 \Rightarrow 2 + 2\tan^2 x + \tan x - 3 = 0$

$\Rightarrow 2\tan^2 x + \tan x - 1 = 0 \Rightarrow (2\tan x - 1)(\tan x + 1) = 0$

$\Rightarrow \tan x = 0.5,\ \tan x = -1$

$\Rightarrow x = 26.6,\ 206.6,\ 135,\ 315$

References: Chapter 8 Pages 187-190

Exercise 8B Q. 1(ii), 2(ii), 3(ii), 4(ii), 7

Compound Angle Formulae

$$\sin(x+y) = \sin x\cos y + \cos x\sin y$$
$$\sin(x-y) = \sin x\cos y - \cos x\sin y$$
$$\cos(x+y) = \cos x\cos y - \sin x\sin y$$
$$\cos(x-y) = \cos x\cos y + \sin x\sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

E.g. $\sin(60^0 - 30^0) = \sin 60^0\cos 30^0 - \cos 60^0\sin 30^0$

$= \frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} - \frac{1}{2}.\frac{1}{2} = \frac{3}{4} - \frac{1}{4} = \frac{1}{2}$

E.g. $\sin(x - 90^0) = \sin x\cos 90^0 - \cos x\sin 90^0 = -\cos x$

E.g. $\cos(60^0 - 30^0) = \cos 60^0\cos 30^0 + \sin 60^0\sin 30^0$

$= \frac{1}{2}.\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}.\frac{1}{2} = \frac{\sqrt{3}}{2}$

References: Chapter 8 Pages 192-196

Exercise 8C Q. 1(ii), 2(ii), 3, 4, 7

Double Angle Formulae

$$\sin 2x = 2\sin x\cos x$$
$$\cos 2x = \cos^2 x - \sin^2 x$$
$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Given, from C2, $\sin^2 x + \cos^2 x = 1$

$\cos 2x = \cos^2 x - \sin^2 x$

$\Rightarrow \cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

Conversely:

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

E.g. show that $\frac{1 - \cos 2x}{\sin 2x} = \tan x$

$\frac{1 - \cos 2x}{\sin 2x} = \frac{2\sin^2 x}{2\sin x\cos x} = \frac{\sin x}{\cos x} = \tan x$

References: Chapter 8 Pages 201-204

Exercise 8E Q. 1(ii), 2(ii), 4(ii), 7

Exercise 8F Q. 2(i), 3(i), 4(i), (vi)

The forms $r\cos(x \pm y)$, $r\sin(x \pm y)$

The expression $a\cos x \pm b\sin x$ can be written in the form $r\cos(x \pm y)$ or $r\sin(x \pm y)$.

In particular:

$$a\cos x + b\sin x = r\left(\frac{a}{r}\cos x + \frac{b}{r}\sin x\right)$$
$$= r\cos(x - \alpha)$$

where $\frac{a}{r} = \cos\alpha,\ \frac{b}{r} = \sin\alpha$

$\Rightarrow r = \sqrt{a^2 + b^2},\ \tan\alpha = \frac{b}{a}$

E.g. Solve the equation $12\cos x + 5\sin x = 4$

$12\cos x + 5\sin x = R\cos(x - \alpha) = R(\cos x\cos\alpha + \sin x\sin\alpha)$

$\Rightarrow R\cos\alpha = 12$ and $R\sin\alpha = 5$

$\Rightarrow R = \sqrt{12^2 + 5^2} = 13,\ \tan\alpha = \frac{5}{12} \Rightarrow \alpha = 22.62^0$

$\Rightarrow 13\cos(x - 22.62^0) = 4 \Rightarrow x - 22.62^0 = \cos^{-1}\frac{4}{13} = 72.08^0$

$\Rightarrow x = 72.08^0 + 22.62^0 = 94.7^0$

Also, $x = 360^0 - 72.08^0 + 22.62^0 = 310.54^0$

E.g. Solve the equation $\sin 2x + 3\cos^3 x = 3\cos x$

$2\sin x\cos x + 3\cos^3 x = 3\cos x$

$\Rightarrow \cos x = 0$ or $2\sin x + 3\cos^2 x = 3$

$\Rightarrow 2\sin x + 3(1 - \sin^2 x) = 3$

$\Rightarrow 2\sin x - 3\sin^2 x = 0$

$\Rightarrow \sin x = 0$ or $\sin x = \frac{2}{3}$

$\Rightarrow x = 0,\ 41.8,\ 90,\ 138.2,\ 180,\ 270,\ 360$

References: Chapter 9 Pages 224-226

Exercise 9A Q. 4

References: Chapter 9 Pages 227-231

Exercise 9A Q. 1(i),(ii), 2, 3

Parametric Equations give a relationship between variables x and y in terms of a third variable, a *parameter*, usually t or θ.
The parametric equations $x = f(t)$, $y = g(t)$ can be written in coordinate form $(f(t), g(t))$.

Graphs of parametric functions can be plotted by substituting values for the parameter.

The **cartesian equation** can be obtained by eliminating the parameter.
This is usually done by making the parameter the subject of one equation and substituting in the other.
In trigonometric parametric equations we can use the identity

$$\sin^2 x + \cos^2 x = 1$$

References: Chapter 9 Pages 231-234

Parametric Equations of standard curves.

The Circle: $x = r\cos\theta,\ y = r\sin\theta \Rightarrow x^2 + y^2 = r^2$

$x = a + r\cos\theta,\ y = b + r\sin\theta \Rightarrow (x-a)^2 + (y-b)^2 = r^2$

The Ellipse: $x = a\cos\theta,\ y = b\sin\theta \Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$

The Parabola: $x = at^2 \quad y = 2at \Rightarrow t = \frac{y}{2a}$

$$\Rightarrow x = a\left(\frac{y}{2a}\right)^2 \Rightarrow y^2 = 4ax$$

The Rectangular Hyperbola: $x = at,\ y = \frac{a}{t} \Rightarrow xy = a^2$

References: Chapter 9 Pages 238-241

Exercise 9B Q. 1(i),(iv), (v), 4

Differentiation of parametric functions

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

So if $x = f(t),\ y = g(t),\ \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$

To find turning points, $\frac{dy}{dx} = 0$

To find the nature of turning points, find the sign of $\frac{d^2y}{dx^2}$,

where $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

Pure Mathematics C4
Version B: page 4
Competence statements g1, g2, g3, g4

E.g. Sketch the graph of $x = 4(t - 1)$, $y = 2t^2$.
Method 1. Substitute values for the parameter.
Plot y against x.

t	0	1	2	3	4	5	6	7
y	0	2	8	18	32	50	72	98
x	-4	0	4	8	12	16	20	24

Method 2. Convert parametric equations to cartesian

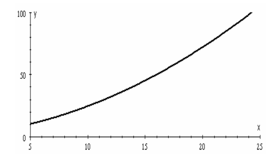
$$x = 4(t-1) \Rightarrow t = \frac{x}{4} + 1$$

Substitute this into the other equation:

$$y = 2\left(\frac{x}{4} + 1\right)^2$$

Create a table of values.
Plot y against x.

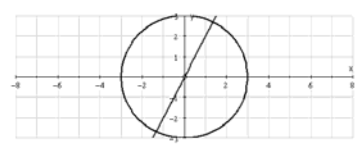
x	-4	0	4	8	12	16	20	24
y	0	2	8	18	32	50	72	98



E.g. Find where the straight line $y = 2x$ intersects the circle $x = 3\cos\theta$, $y = 3\sin\theta$.
Substituting $x = 3\cos\theta$, $y = 3\sin\theta$ into $y = 2x$ gives

$$3\sin\theta = 6\cos\theta \Rightarrow \tan\theta = 2 \Rightarrow \theta = 63.4^0,\ 243.4^0.$$

Substituting these in $x = 3\cos\theta$, $y = 3\sin\theta$ gives the intersection points (1.34, 2.68) and (-1.34, -2.68)



E.g. Find and classify the nature of the stationary point on the curve $x = 4 - t^3$, $y = t^2 - 2t$.

$x = 4 - t^3 \Rightarrow \frac{dx}{dt} = -3t^2$ and $y = t^2 - 2t \Rightarrow \frac{dy}{dt} = 2t - 2$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \Rightarrow \frac{dy}{dx} = \frac{2t-2}{-3t^2} = \frac{2-2t}{3t^2}$$

At stationary point $\frac{dy}{dx} = 0 \Rightarrow \frac{2-2t}{3t^2} = 0 \Rightarrow t = 1$

Substitute in the parametric equations gives the stationary point (3, -1).

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \Rightarrow \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{2-2t}{3t^2}\right)}{-3t^2} = \frac{2(2-t)}{9t^5}$$

Substitute $t = 1$ gives $\frac{d^2y}{dx^2} = \frac{2}{9} > 0$ so minimum.

References:
Chapter 10
Pages 253-258

Exercise 10A
Q. 2(iii), 3

Volume of a solid of revolution
For a solid that is formed by revolution of a curve $y = f(x)$ through 360^0 about the x-axis:

$$V = \int_a^b \pi y^2 \, dx$$

For a solid that is formed by revolution of a curve $x = f(y)$ through 360^0 about the y axis:

$$V = \int_a^b \pi x^2 \, dy$$

E.g. The curve $y^2 = x + 1$ is rotated about the x-axis. Find the volume of this solid between $x = -1$ and $x = 1$.

$$V = \int_{-1}^{1} \pi y^2 \, dx = \pi \int_{-1}^{1} (x+1) \, dx = \pi \left[\frac{x^2}{2} + x \right]_{-1}^{1}$$
$$= \pi\left(\frac{1}{2}+1\right) - \pi\left(\frac{1}{2}-1\right) = 2\pi$$

References:
Chapter 10
Pages 261-263

Exercise 10B
Q. 1(i),(ii),
(iii), 5

Partial Fractions
The process of partial fractions may be used to carry out an integration of an algebraic fraction.

E.g. $\displaystyle\int \frac{1}{(x-a)(x-b)} dx = \int \left(\frac{A}{x-a}\right) + \left(\frac{B}{x-b}\right) dx$

$= A\ln(x-a) + B\ln(x-b)$

E.g. $\displaystyle\int \frac{A}{(x-a)^2(x-b)} dx = \int \left(\frac{B}{(x-a)^2} + \frac{C}{(x-a)} + \frac{D}{(x-b)} \right) dx$

$\displaystyle = C\ln(x-a) + D\ln(x-b) - \frac{B}{(x-a)}$

E.g. $\displaystyle\int \frac{A}{(x^2+a)(x-b)} dx = \int \left(\frac{Bx}{(x^2+a)} + \frac{C}{(x-b)} \right) dx$

$\displaystyle = \frac{B}{2}\ln(x^2+a) + c\ln(x-b)$

N.B. If you are asked to do such an integration the numerator will always come out as Bx in this module. If the x were not there the integral would be outside the scope of this module.

E.g. Find $\displaystyle\int_1^2 \frac{x-1}{(1+x)(1+x^2)} dx$

$$\frac{x-1}{(1+x)(1+x^2)} = \frac{A}{1+x} + \frac{B+Cx}{1+x^2}$$
$\Rightarrow x - 1 \equiv A(1+x^2) + (1+x)(B+Cx)$

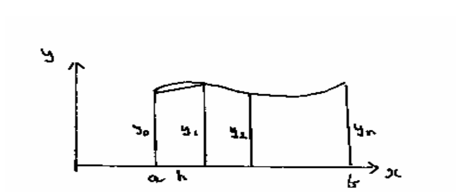
Compare coefficients:

$A + C = 0,\ B + C = 1,\ A + B = -1$

$\Rightarrow A = -1,\ B = 0,\ C = 1$

$$\Rightarrow \int_1^2 \frac{x-1}{(1+x)(1+x^2)} dx = \int_1^2 \left(\frac{-1}{1+x} + \frac{x}{1+x^2} \right) dx$$
$$= \left[-\ln(1+x) + \frac{1}{2}\ln(1+x^2) \right]_1^2$$
$$= \left(\frac{1}{2}\ln 5 - \ln 3\right) - \left(\frac{1}{2}\ln 2 - \ln 2\right)$$
$$= \left(\frac{1}{2}(\ln 5 + \ln 2) - \ln 3\right) = \left(\frac{1}{2}\ln 10 - \ln 3\right) \approx -1.33 \text{ (3 s.f.)}$$

Trapezium Rule
In C2 you met the use of the Trapezium Rule to estimate the area under a curve.



$$\int_a^b y\,dx \approx \frac{h}{2}\left[y_0 + 2(y_1 + y_2 + ... + y_{n-1}) + y_n \right] \text{ where } h = \frac{b-a}{n}$$

In this module you will be expected to apply the rule for an increasing number of strips and comment on the accuracy of the estimates.

E.g. Estimate $\displaystyle\int_0^1 \frac{1}{1+x^2} dx$ using the trapezium rule with (i) 2 strips, (ii) 4 strips and (iii) 8 strips.

x	y	2 strips	4 strips	8 strips
0	1	1	1	1
0125	0.9846			1.9692
0.25	0.9412		1.8824	1.8824
0.375	0.8767			1.7534
0.5	0.8	1.6	1.6	1.6
0.625	0.7191			1.4382
0.75	0.64		1.28	1.28
0.875	0.5664			1.1327
1	0.5	0.5	0.5	0.5
	SUM	3.1	6.2624	12.556
	SUM*h/2	0.775	0.7828	0.7847

It can be seen that $T_2 = 0.775$, $T_4 = 0.78279$ and $T_8 = 0.78474$.
It could be assumed that T = 0.785.

References:
Chapter 11
Pages 275-280

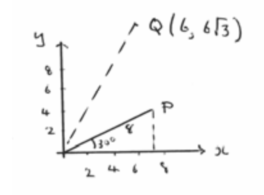
Exercise 11A
Q. 1(ii),2(ii),
3(ii),4(ii)

Terminology

A **Vector** quantity has magnitude and direction.
A **Scalar** quantity has only magnitude.
On a vector diagram, a vector is represented as an arrowed line.
The length of the line is the magnitude and the direction is indicated by the line and arrowhead.
The direction is usually the angle measured anticlockwise from the positive x-axis.
In polar form this vector is given as (r, θ).
In component form, the vector is (a, b), which can also be written as $a\mathbf{i} + b\mathbf{j}$ or $\begin{pmatrix} a \\ b \end{pmatrix}$

The description of the vector does not include its position.
The **position vector** $\overrightarrow{OP}$ of a point P is the vector from O to P.

$\overrightarrow{OP} = (8, 30^0)$ in polar form $= 8\cos 30\mathbf{i} + 8\sin 30\mathbf{j}$ $= 4\sqrt{3}\,\mathbf{i} + 4\mathbf{j}$



Q has position vector $\mathbf{q} = \overrightarrow{OQ} = 6\,\mathbf{i} + 6\sqrt{3}\mathbf{j}$

$|\mathbf{q}| = \sqrt{6^2 + 6^2.3} = 12$

and $\mathbf{q}$ makes an angle θ with the x-axis

where $\tan\theta = \frac{6\sqrt{3}}{6} \Rightarrow \theta = 60^0$.

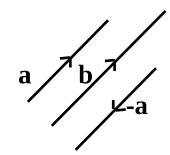
Hence, in polar form, $\mathbf{q} = (12, 60^0)$

References:
Chapter 11
Pages 282-283

Multiplication by a scalar

$$\lambda\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} \qquad -\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -a \\ -b \end{pmatrix}$$

E.g. $\mathbf{a} = 3\mathbf{i} + 4\mathbf{j}$,
$\mathbf{b} = 2\mathbf{a} = 6\mathbf{i} + 8\mathbf{j}$
$\mathbf{c} = -\mathbf{a} = -3\mathbf{i} - 4\mathbf{j}$



References:
Chapter 11
Pages 283-285

Exercise 11B
Q. 1(ii),2(ii),4

Adding and subtracting Vectors

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$$

$(a\mathbf{i} + b\mathbf{j}) + (c\mathbf{i} + d\mathbf{j}) = (a+c)\mathbf{i} + (b+d)\mathbf{j}$

Subtracting vectors is the same as adding the negative of the vector.

E.g. $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{b} = 3\mathbf{i} - 5\mathbf{j}$
$\mathbf{a} + \mathbf{b} = 5\mathbf{i} - 4\mathbf{j}$,
$\mathbf{a} - \mathbf{b} = -\mathbf{i} + 6\mathbf{j}$

E.g. Find k such that $\mathbf{a} + k\mathbf{b}$ is parallel to the x-axis.
$\mathbf{a} + k\mathbf{b} = 2\mathbf{i} + \mathbf{j} + k(3\mathbf{i} - 5\mathbf{j}) = (2+3k)\mathbf{i} + (1-5k)\mathbf{j}$
When $k = {}^1/_5$, $\mathbf{a} + k\mathbf{b} = {}^{13}/_5\,\mathbf{i}$

References:
Chapter 11
Page 287

Exercise 11B
Q. 6(i),(ii)

The Unit Vector

A unit vector is a vector with magnitude 1.

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|a|}$$

E.g. $\mathbf{a} = 2\mathbf{i} + \mathbf{j} \Rightarrow |\mathbf{a}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\Rightarrow \hat{\mathbf{a}} = \frac{\mathbf{a}}{|a|} = \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{5}}$

References:
Chapter 11
Pages 291-297

Exercise 11C
Q. 1(ii),2(ii),
4(ii), 5(ii)

Coordinate Geometry in 2 dimensions

The equation of the line through (a,b) in direction $\begin{pmatrix} c \\ d \end{pmatrix}$ is

$$\mathbf{r} = \begin{pmatrix} a \\ b \end{pmatrix} + \lambda\begin{pmatrix} c \\ d \end{pmatrix}$$

This can be written

$$\frac{x-a}{c} = \frac{y-b}{d} \text{ providing } c, d \neq 0$$

E.g. Find the vector equation of the line passing through the points (1, 3) and (5, 8).

The direction of the line is $\begin{pmatrix} 5 \\ 8 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\Rightarrow r = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 5 \end{pmatrix} \Rightarrow \frac{x-1}{4} = \frac{y-3}{5}$

E.g. Find the point of intersection of the lines

$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \lambda\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 6 \\ 1 \end{pmatrix} + \kappa\begin{pmatrix} 3 \\ 8 \end{pmatrix}$

The same point satisfies both equations
$\Rightarrow 1 + 4\lambda = 6 + 3\kappa$ and $3 + 3\lambda = 1 + 8\kappa$
$\Rightarrow 4\lambda - 3\kappa = 5, \quad 3\lambda - 8\kappa = -2$
Solve simultaneously $\Rightarrow \lambda = 2, \kappa = 1$

$\Rightarrow$ point of intersection is $\begin{pmatrix} 9 \\ 9 \end{pmatrix}$

Summary C4 Topic 5: **Vectors - 2**

References:
Chapter 11
Pages 299-301

Exercise 11D
Q. 1(ii),(v), 4

The Scalar Product and angle between two lines

If $\overrightarrow{OA} = \mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $\overrightarrow{OB} = \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

then $|OA| = \sqrt{a_1^2 + a_2^2}$ and $|OB| = \sqrt{b_1^2 + b_2^2}$

The angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$ is given by

$\cos\theta = \dfrac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|}$ where $\mathbf{a.b} = a_1 b_1 + a_2 b_2$

If the vectors are perpendicular then **a.b** = 0

E.g. Find the angle between the lines:

$$r = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

$$\cos\vartheta = \frac{\begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}}{\sqrt{(2^2 + 0^2 + (-1)^2) \cdot (1^2 + 1^2 + 3^2)}} = \frac{-1}{\sqrt{55}}$$

and thus the angle = 97.7^0.

References:
Chapter 11
Pages 303-311

Exercise 11E
Q. 1(ii),2(ii),
3(ii), 8

Coordinate Geometry in 3 dimensions

A vector in 3-D is given by $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

The equation of a straight line through (a, b, c) with direction $\begin{pmatrix} d \\ e \\ f \end{pmatrix}$ is given by $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

or $\dfrac{x-a}{d} = \dfrac{y-b}{e} = \dfrac{z-c}{f}$

E.g. Find the equation of the plane perpendicular to $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and passing through the point (1, −1. 0).

The plane is $r \bullet \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = 1 - 2 = -1 \Rightarrow r \bullet \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = -1$

or $x + 2y + z = -1$

References:
Chapter 11
Pages 315-319

Planes

If the perpendicular direction is $\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix}$ then the equation of the plane is

$$n_1 x + n_2 y + n_3 z + d = 0$$

where $d = -\mathbf{a.n}$ and **a** is the position vector of a point on the plane.

E.g. Find the equation of the plane through A (1, 3, 1), B (1, 2, 4) and C (2, 3, 6)

Let the normal vector be given by $\mathbf{n} = \begin{pmatrix} a \\ b \\ 1 \end{pmatrix}$. The scalar product of **n** with $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are 0. $\overrightarrow{AB} = \begin{pmatrix} 0 \\ -1 \\ 3 \end{pmatrix}, \Rightarrow \mathbf{n} \bullet \overrightarrow{AB} = 0$

$\Rightarrow 0 - b + 3 = 0 \Rightarrow b = 3$. Similarly $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix} \Rightarrow \mathbf{n} \bullet \overrightarrow{AC} = 0$

$\Rightarrow a + 5 = 0 \Rightarrow a = -5 \Rightarrow r \bullet \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} = 5 \Rightarrow$ or $-5x + 3y + z = 5$

References:
Chapter 11
Pages 320-322

Intersection of a plane and a line

Find the parametric form of the line and substitute into the plane.

e.g. $\mathbf{r} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \lambda \begin{pmatrix} d \\ e \\ f \end{pmatrix} = \begin{pmatrix} a + \lambda d \\ b + \lambda e \\ c + \lambda f \end{pmatrix}$ into $n_1 x + n_2 y + n_3 z + d = 0$

will give an equation in λ which can be solved.

E.g. Find the intersection of the line $\dfrac{x-5}{-5} = \dfrac{y+2}{3} = \dfrac{z-1}{1}$ and the plane $-5x + 3y + z = 5$

The line is $r = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} \Rightarrow x = 5 - 5\lambda,\ y = -2 + 3\lambda,\ z = 1 + \lambda$

Substitute into the plane: $\Rightarrow -5(5 - 5\lambda) + 3(-2 + 3\lambda) + (1 + \lambda) = 5$

$\Rightarrow \lambda = 1 \Rightarrow$ intersection is (0, 1, 2).

Exercise 11F
Q. 2, 12

Distance of a point from a plane

- Construct the line through the given point in the perpendicular direction of the plane.
- Find where this line cuts the plane.
- Find the distance between the two points.

E.g. Find the distance of (1, 2, 3) from the plane $2x + 3y + z = 4$.
The perpendicular line through (1, 2, 3) cuts the plane at the point (0, 0.5, 2.5).
The distance between this point and (1, 2, 3) is $\sqrt{\dfrac{7}{2}}$.

References: Chapter 12 Pages 335-336

Differential Equations

A first order differential equation contains a derivative such as $\frac{\mathrm{d}y}{\mathrm{d}x}$.

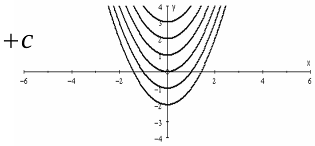
Exercise 12A Q. 1

The general solution is the equation of the family of curves satisfying the differential equation. It involves a constant of integration.

References: Chapter 12 Pages 336-339

A particular solution is a single member of the family of curves. A single piece of information (e.g. a point through which it goes) is required to determine the constant.

E.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \Rightarrow y = x^2 + c$



If the curve passes through (1, 2)
then $2 = 1 + c \Rightarrow y = x^2 + 1$
Similarly, if the curve passes through (1, 5)
then $5 = 1 + c \Rightarrow y = x^2 + 4$

Differential equations are equations which involve rates of change. (Rate of change usually means with respect to time, unless otherwise specified.

Exercise 12A Q. 3, 10, 12

i.e. the rate at which P changes is $\frac{\mathrm{d}P}{\mathrm{d}t}$.

A decreasing rate is indicated by a negative sign.

E.g. A bacteria population increases at a rate proportional to the current population.
When P = 60 million, the rate of increase is 2 million per hour.

$\frac{\mathrm{d}P}{\mathrm{d}t} = kP$. When $P = 60$, $\frac{\mathrm{d}P}{\mathrm{d}t} = 2 \Rightarrow k = \frac{1}{30}$

$\Rightarrow \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{P}{30}$.

References: Chapter 12 Pages 341-342

The general solution of a differential equation

The general solution is a whole family of curves as integration is performed and therefore an arbitrary constant is involved.

e.g. $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \Rightarrow y = x^2 + c.$

E.g. $y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x \Rightarrow \int y\,\mathrm{d}y = \int 2x\,\mathrm{d}x$

$\Rightarrow \frac{y^2}{2} = x^2 + c' \Rightarrow y^2 = 2x^2 + c$

Separation of variables

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}(x)\mathrm{g}(y) \Rightarrow \int \frac{\mathrm{d}y}{\mathrm{g}(y)} = \int \mathrm{f}(x)\,\mathrm{d}x.$$

E.g. $\mathrm{e}^x\frac{\mathrm{d}y}{\mathrm{d}x} = y \Rightarrow \int \frac{\mathrm{d}y}{y} = \int \mathrm{e}^{-x}\,\mathrm{d}x$

$\Rightarrow \ln y = -\mathrm{e}^{-x} + c$

- We treat $\frac{\mathrm{d}y}{\mathrm{d}x}$ as a fraction
- Gather x terms with dx and y terms with dy
- Integrate both sides
- Include only one arbitrary constant of integration
- When logs are involved, the constant, c, can be written $\ln A$ and be absorbed into the log

Exercise 12B Q. 2(i),(iii),(v)

E.g. Find the solution to the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$$

given that when $x = 2, y = 3$

$x\frac{\mathrm{d}y}{\mathrm{d}x} = 2y \Rightarrow \int \frac{\mathrm{d}y}{y} = 2\int \frac{\mathrm{d}x}{x} \Rightarrow \ln y = 2\ln x + \ln A = \ln Ax^2$

$\Rightarrow y = Ax^2$

When $x = 2, y = 3 \Rightarrow 3 = 4A \Rightarrow A = \frac{3}{4} \Rightarrow y = \frac{3}{4}x^2$

References: Chapter 12 Pages 344-348

Particular solutions

The "piece of information" required to find a particular solution is usually a point through which the curve passes.

Exercise 12C Q. 1(iii),(v), 5