REVISION SHEET – FP2 (AQA)

CALCULUS

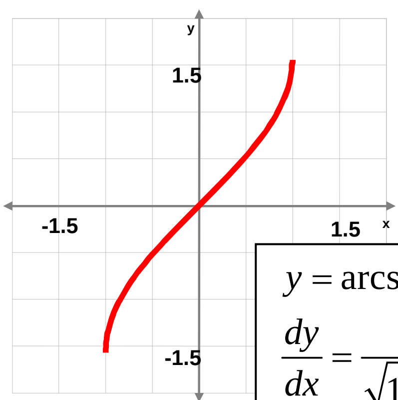
The main ideas are:

- Calculus using inverse trig functions & hyperbolic trig functions and their inverses.
- Calculating arc lengths.

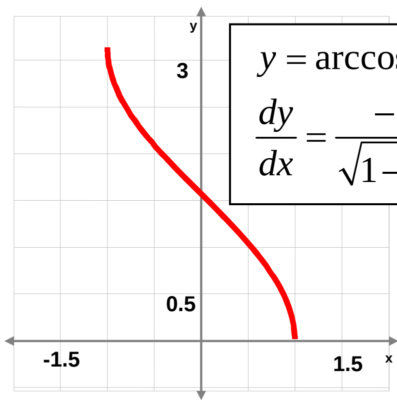
Before the exam you should know:

- That you can differentiate the trig functions, the hyperbolic trig functions and their inverses.
- That you can apply the standard rules for differentiation (product rule, quotient rule and chain rule) to functions which involve the above.
- That you can integrate trig functions and hyperbolic trig functions.
- That you can integrate, arcsin(x), arccos(x), arctan(x), arccot(x), arsinh(x), arcosh(x) etc using integration by parts.
- Your trig identities and hyperbolic function identities and how to use them in integration problems. Particularly get familiar with useful substitutions to make for certain problems.
- How to calculate arc lengths.

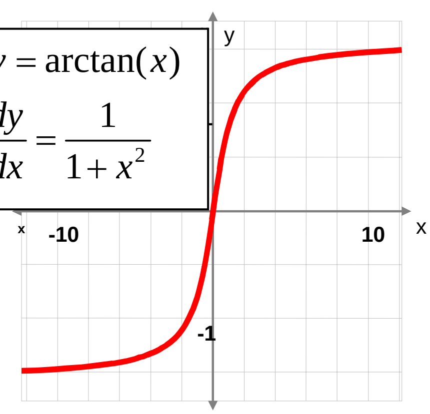
Differentiating the Inverse Trig Functions



$$y = \arcsin(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$



$$y = \arccos(x) \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$



$$y = \arctan(x) \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$

It is important to be aware of what the range is for each of these, namely:

$$-\frac{\pi}{2} \le \arcsin \le \frac{\pi}{2}\ , \qquad 0 \le \arccos \le \pi\ , \qquad -\frac{\pi}{2} \le \arctan \le \frac{\pi}{2}$$

Standard Calculus of Inverse Trig and Hyperbolic Trig Functions

$$y = \arcsin(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$y = \arccos(x) \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$y = \arctan(x) \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y = \operatorname{ars\,inh}(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$y = \operatorname{ar\,cosh}(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}} = \operatorname{ar\,cosh}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2+a^2}} = \operatorname{ar\,sinh}\left(\frac{x}{a}\right) + c$$

Calculus using these functions

The examples below are very typical and show most of the common tricks. Note – details of all substitutions have been omitted, make sure you understand how to do them in this case and also in the case of a definite integral.

- $\int \frac{1}{\sqrt{4x^2+16x+32}}dx = \frac{1}{2}\int \frac{1}{\sqrt{(x+2)^2+4}}dx = \frac{1}{2}\text{arsinh}\left(\frac{x+2}{2}\right)+c$
- $\int \frac{4}{\sqrt{5+3x-9x^2}}dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{5}{9}-\left(x^2-\frac{x}{3}\right)}}dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{21}{36}-\left(x-\frac{1}{6}\right)^2}}dx = \frac{4}{3}\arcsin\left(\frac{6\left(x-\frac{1}{6}\right)}{\sqrt{21}}\right)+c = \frac{4}{3}\arcsin\left(\frac{6x-1}{\sqrt{21}}\right)+c$
- $\int \frac{3}{\sqrt{2x^2+4x-10}}dx = \frac{3}{\sqrt{2}}\int \frac{1}{\sqrt{(x+1)^2-6}}dx = \frac{3}{\sqrt{2}}\text{arcosh}\left(\frac{x+1}{\sqrt{6}}\right)+c$
- $y = \text{arcosh}(x^2) \Rightarrow \frac{dy}{dx} = \frac{2x}{\sqrt{x^4-1}}$ (to see this use the chain rule, set $z = x^2$ and then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$).

Some useful integration tricks

Splitting up an integration: e.g. $\int_1^5 \frac{x+5}{x^2+4}dx = \int_1^5 \frac{x}{x^2+4}dx + \int_1^5 \frac{5}{x^2+4}dx$

By inspection: e.g. Since $\ln(x^2+4)$ gives $\frac{2x}{x^2+4}$ when differentiated, we have $\int \frac{x}{x^2+4}dx = \frac{1}{2}\ln(x^2+4)+c$ or since $(x^2+1)^{\frac{1}{2}}$ gives $x(x^2+1)^{-\frac{1}{2}}$ when differentiated, we have $\int \frac{x}{\sqrt{x^2+1}}dx = \sqrt{x^2+1}+c$

Using clever substitutions: e.g. the substitution $u = \sinh(x)$ will help you with $\int \sqrt{x^2+1}dx$.

Arc Length and Area

The length of an arc between points A and B on a curve can be calculated by

$$\int_{x_A}^{x_B}\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx \text{ or } \int_{y_A}^{y_B}\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}dy$$

In parametric form this is: $\int_{t_A}^{t_B}\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}}dt$

The area of the surface formed when arc AB is rotated completely about Ox is

$$2\pi\int_{x_A}^{x_B} y\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{1}{2}}dx \text{ or } 2\pi\int_{y_A}^{y_B} x\left[1+\left(\frac{dx}{dy}\right)^2\right]^{\frac{1}{2}}dy$$

In parametric form this (when rotated about the Ox) is: $2\pi\int_{t_A}^{t_B} y\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{\frac{1}{2}}dt$

You should review examples of how this type of question and how to solve them. This obviously involves differentiation, algebraic manipulation and integration (often by substitution).

REVISION SHEET – FP2 (AQA)

COMPLEX NUMBERS 1

The main ideas are:

- Manipulating complex numbers
- Complex conjugates and roots of equations
- The Argand diagram
- Multiplying and dividing in polar form

Before the exam you should know:

- Multiply two complex numbers quickly and in one step, this will save you a lot of time in the exam.
- Geometrically interpret $|z_1 - z_2|$ as the distance between the complex numbers z_1 and z_2 in the Argand diagram.
- Use the fact that $|z_1 + z_2| = |z_1 - (-z_2)|$ which equals the distance between z_1 and $-z_2$ in the Argand diagram.
- Remember the exact values of the sine and cosine angles which are multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$, eg $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

Manipulating Complex Numbers.

Multiplying, dividing, adding and subtracting

- Multiplying, adding and subtracting were all covered in material in FP1.
- You are also now required to be able to divide complex numbers, which is slightly more complicated. Whenever you see a complex number on the denominator of a fraction you can “get rid of it” by multiplying both top and bottom of the fraction by its complex conjugate.

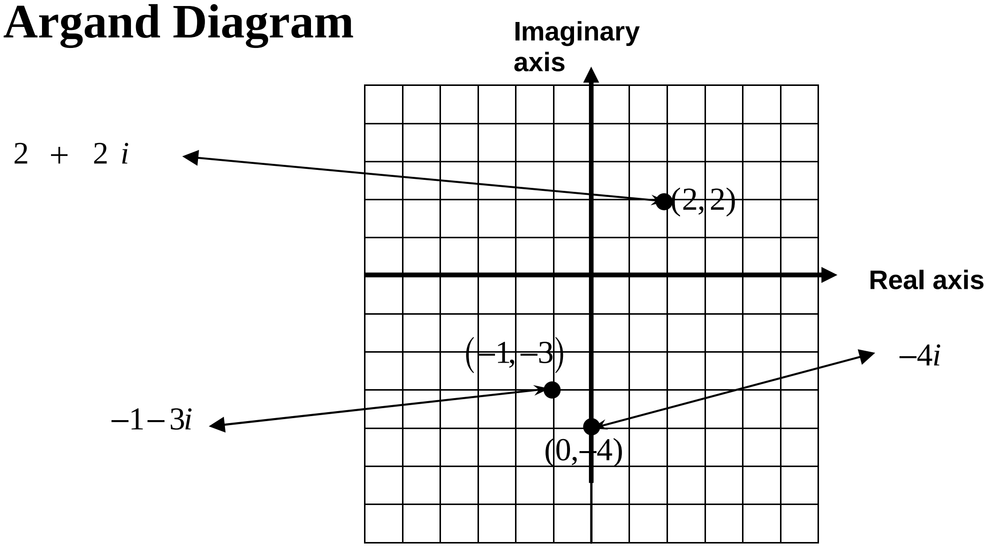
$$\text{e.g. } \frac{3+2i}{1-i} = \left(\frac{3+2i}{1-i}\right)\left(\frac{1+i}{1+i}\right) = \frac{1+5i}{2}$$

Complex Conjugates and Roots of Equations

The complex conjugate of $z = a + bi$ is $z* = a - bi$.

- Remember $zz*$ is a real number and it equals the square of the modulus of z.
- Complex roots of polynomial equations with real coefficients occur in conjugate pairs. This means that if you are told one complex root of a polynomial equation with real coefficients you are in fact being told two roots, two for the price of one). This is key to answering some very typical exam questions.
- Due to the above, a polynomial equation with real coefficients of odd degree must have at least one real root. In certain exam questions you must use this fact to your advantage.

The Argand Diagram



- In the Argand diagram the point (x, y) corresponds to the complex number $x + yi$.
- You should be aware that the set of complex numbers z with for example $|z-5+i|=6$ is a circle of radius 6 centred at $5-i$ (or (5, –1)) in the Argand plane.
- The argument of a complex number z, denoted $\arg(z)$ is the angle it makes with the positive real axis in the Argand diagram, measured anticlockwise and such that $-\pi < \arg(z) \le \pi$.
- When answering exam questions about points in the Argand diagram be prepared to used geometrical arguments based around equilateral triangles, similar triangles, isosceles triangles and parallel lines to calculate lengths and angles.

Other sets of points in the complex plane.

Where *a* and *b* are complex numbers, the set of complex numbers *z* such that

1. $\arg(z-a)=\theta$, is a half line starting from *a* in the direction θ
2. $\arg(z-a)=\arg(z-b)$, is the line through *a* and *b* with the section between *a* and *b* (inclusive) removed.
3. $\arg(z-a)=\arg(z-b)+\pi$, is the line from *a* to *b* (not including *a* and *b* themselves).

Multiplying and Dividing in Polar Form

- If $z=x+yi$ has $|z|=r$ and $\arg(z)=\theta$ then $z=r(\cos\theta+i\sin\theta)$. This is called the *polar* or *modulus-argument* form.
- To multiply complex numbers in polar form we multiply their moduli and add their arguments. So if z_1 and z_2 are complex numbers we have $|z_1z_2|=|z_1||z_2|$ and $\arg(z_1z_2)=\arg z_1+\arg z_2$. Note: you may have to make adjustments so that $\arg(z_1z_2)$ is in the required range for example if $\arg z_1=\frac{7\pi}{12}$ and $\arg z_2=\frac{\pi}{2}$ then

$$\arg(z_1z_2)=\arg z_1+\arg z_2=\frac{7\pi}{12}+\frac{\pi}{2}=\frac{7\pi+6\pi}{12}=\frac{13\pi}{12}\equiv-\frac{11\pi}{12}$$

- To divide complex numbers in polar form we divide their moduli and subtract their arguments. So if z_1 and z_2 are complex numbers we have $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right)=\arg z_1-\arg z_2$. Again, adjustments to $\arg\left(\frac{z_1}{z_2}\right)$ may be necessary.

REVISION SHEET – FP2 (AQA)

COMPLEX NUMBERS 2

The main ideas are:

- De Moivre's Theorem and its applications
- Exponential notation
- Using both of the above to get formulae by summing C+jS series.
- n^{th} roots of complex numbers

Before the exam you should know:

- How to multiply and divide complex numbers in polar form.
- What de Moivre's theorem is and how to apply it.
- About the exponential notation $e^{i\theta} = \cos\theta + i\sin\theta$, $z = x + yi = re^{i\theta}$
- How to apply de Moivre's theorem to finding multiple angle formulae and to summing series.
- About the n *n*th roots of unity, including how to represent them on an Argand diagram.

De Moivre's Theorem

De Moivre's Theorem states that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for any integer *n*. Some applications of this are shown below.

Example 1

Evaluate $(1+i)^{12}$.

Solution

The first thing to do is to write $1+\text{j}$ in polar form.

This is just $1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\text{Therefore } (1+i)^{12} = \left(\sqrt{2}\right)^{12}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{12}$$
$$= 64\left(\cos 3\pi + i\sin 3\pi\right)$$
$$= 64\left(\cos \pi + i\sin \pi\right)$$
$$= 64(-1+0)$$

Note: in example 2 on the right it is typical to be asked to go on to integrate $\sin^6\theta$. De Moivre's theorem can also be used to express multiple angles in terms of powers of the trig functions in a very straightforward way.

Example 2

Express $\sin^6\theta$ in terms of multiple angles.

Solution

If $z = \cos\theta + i\sin\theta$ then $2i\sin\theta = z - z^{-1}$.
So,

$$(2i)^6\sin^6\theta = \left(z - z^{-1}\right)^6$$
$$= z^6 - 6z^5z^{-1} + 15z^4z^{-2} - 20z^3z^{-3} + 15z^2z^{-4} - 6zz^{-5} + z^{-6}$$
$$= z^6 + z^{-6} - 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) - 20$$
$$= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

Therefore,

$$-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

$$\sin^6\theta = \frac{20 - 2\cos 6\theta + 12\cos 4\theta - 30\cos 2\theta}{64}$$
$$= \frac{10 - \cos 6\theta + 6\cos 4\theta - 15\cos 2\theta}{32}.$$

Exponential notation for complex numbers

Exponential notation begins with $e^{i\theta} = \cos\theta + i\sin\theta$. This means that any complex number, *z*, can be written in polar form as $z = x + yi = re^{i\theta}$ where r is the modulus of z and θ is the argument of *z*.

n^th roots of complex numbers

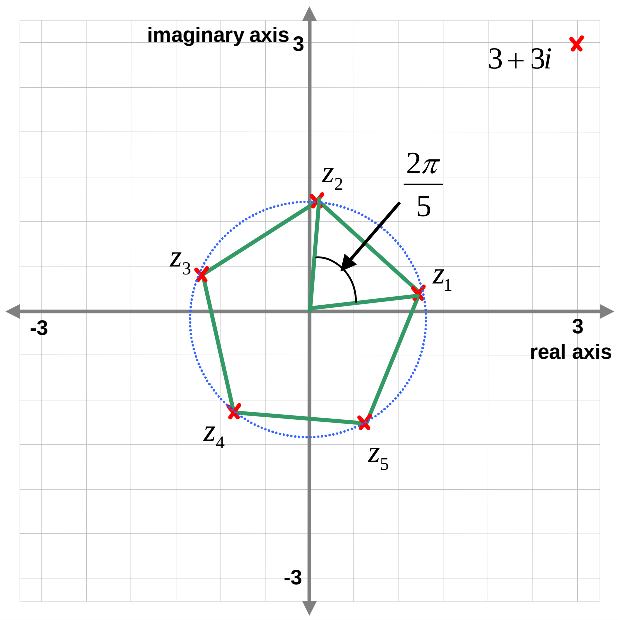
The non-zero complex number $r(\cos\theta + i\sin\theta)$ has *n* different *n*th roots, which are:

$$r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right),$$

where $k = 0, 1, 2, \ldots, n-1$.

nth roots of complex numbers are best thought about geometrically, the diagram shows the 5^{th} roots of 3+3*i*.

You should be able to express these roots in polar form using the exponential notation.



Example

Find all the fourth roots of –64.

Solution

$$-64 = 64\left(\cos(\pi) + i\sin(\pi)\right) = 64e^{i\pi}$$

The modulus of each of the fourth roots must be the positive real fourth root of 64. This is

$$\sqrt[4]{64} = 64^{\frac{1}{4}} = \left(2^6\right)^{\frac{1}{4}} = 2^{\frac{3}{2}} = 2\sqrt{2}$$

The argument of one of the roots is a quarter of the argument of –64. The argument of –64 is π so this is $\frac{\pi}{4}$.

So one of the fourth roots of –64 is $2\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right) = 2\sqrt{2}e^{i\frac{\pi}{4}}$

And the other fourth roots have the same modulus and arguments which are a further $\frac{2\pi}{4} = \frac{\pi}{2}$ "on" from this one. These are therefore,

$$2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)\right) = 2\sqrt{2}e^{i\frac{3\pi}{4}}$$

$$2\sqrt{2}\left(\cos\left(\frac{5\pi}{4}\right)+i\sin\left(\frac{5\pi}{4}\right)\right) = 2\sqrt{2}\left(\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)2\sqrt{2}e^{-i\frac{3\pi}{4}}$$

$$2\sqrt{2}\left(\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)\right) = 2\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)2\sqrt{2}e^{-i\frac{\pi}{4}}$$

REVISION SHEET – FP2 (AQA)

HYPERBOLIC TRIG FUNCTIONS

The main ideas are:

- Definitions of the hyperbolic trig functions and their inverses.
- Working with the hyperbolic trig functions
- Identities involving hyperbolic trig functions

Before the exam you should know:

- The definitions $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$, $\tanh(x)=\dfrac{\sinh(x)}{\cosh(x)}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$
- That you can prove that $\text{arccosh}(x)=\ln(x+\sqrt{x^2-1})$, $\text{arsinh}(x)=\ln(x+\sqrt{x^2+1})$ $\text{artanh}(x)=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$
- Your trig identities and hyperbolic function identities, experience will tell you when it is best to work in the exponential form when dealing with equations.
- And be able to prove hyperbolic identities from the definitions $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$, it's worth practicing indices for this.

The Hyperbolic Trig Functions

These are defined as:

$\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$,

$\tanh(x)=\dfrac{\sinh(x)}{\cosh(x)}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

For example, $\sinh(\ln 10)=\dfrac{e^{\ln 10}-e^{-\ln 10}}{2}=\dfrac{10-\frac{1}{10}}{2}=\dfrac{99}{20}$.

The Inverse Hyperbolic Trig Functions

Just as the hyperbolic trig functions are defined in terms of e^x, their inverses can be expressed in term of logs. In fact $\text{arcosh}(x)=\ln(x+\sqrt{x^2-1})$, $\text{arsinh}(x)=\ln(x+\sqrt{x^2+1})$, $\text{artanh}(x)=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. You should be able to prove (and use) all of these. Here is the proof that $\text{arcosh}(x)=\ln(x+\sqrt{x^2-1})$.

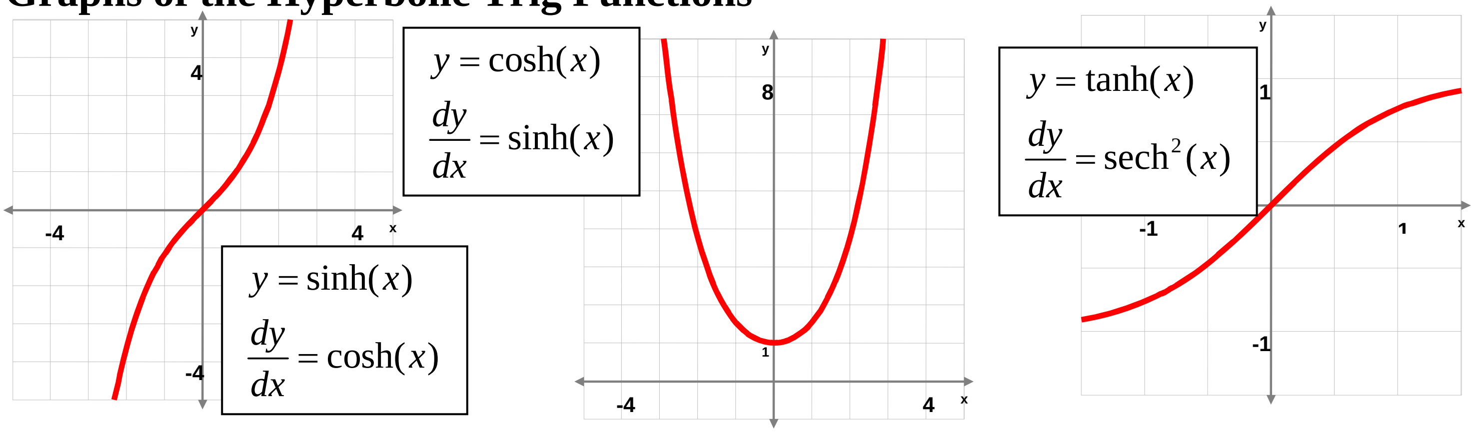
Let $y=\text{ar}\cosh(x)$, then $x=\cosh(y)=\dfrac{e^y+e^{-y}}{2}$. Rearranging this gives $0=e^y-2x+e^{-y}$. Multiplying this by e^y gives $0=e^{2y}-2xe^y+1$. This is a quadratic in e^y and using the formula for the roots of a quadratic gives $e^y=\dfrac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2-1}$. Taking logs gives $y=\text{ar}\cosh(x)=\ln(x\pm\sqrt{x^2-1})$. Do you know why the expression with the minus sign is rejected here?

These expressions can be used to give exact values of the inverse hyperbolic trig functions in term of logs.

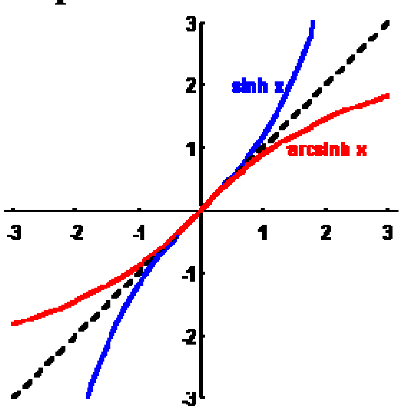
For example,

$$\text{arcosh}\left(\frac{5}{3}\right)=\ln\left(\frac{5}{3}+\sqrt{\left(\frac{5}{3}\right)^2-1}\right)=\ln\left(\frac{5}{3}+\sqrt{\frac{16}{9}}\right)=\ln(3).$$

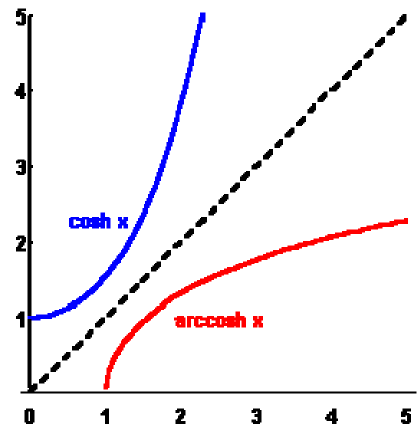
Graphs of the Hyperbolic Trig Functions



Graphs of the Inverse Hyperbolic Trig Functions



You must also know the graphs of the inverse hyperbolic trig functions, arsinh, arcosh and artanh. As for any function these are obtained by reflecting the respective graphs of sinh, cosh and tanh in $y = x$. The examples of arsinh and arcosh are shown here. Notice that arcosh(x) is only defined for x greater than or equal to 1.



Identities Involving Hyperbolic Trig Functions

Identities involving hyperbolic trig functions include:

$$\cosh^2 u - \sinh^2 u = 1, \quad \cosh(2u) = \cosh^2 u + \sinh^2 u, \quad \sin(u+v) = \sinh(u)\cosh(v) + \cosh(u)\sinh(v)$$

The only difference between a hyperbolic trig identity and the corresponding standard trig identity is that the sign is reversed when a product of two sines is replaced by a product of two sinhs. This is called Osborn's Rule.

You can prove any hyperbolic trig identity using their definitions and should be able to do this for the exam.

Equations Involving Hyperbolic Trig Functions

Example

Solve the equation $13\cosh x + 5\sinh x = 20$ giving your answer in terms of natural logarithms.

Solution $$13\cosh x + 5\sinh x = 20 \Rightarrow 13\left(\frac{e^x + e^{-x}}{2}\right) + 5\left(\frac{e^x - e^{-x}}{2}\right) = 20$$

$$\Rightarrow 18e^x + 8e^{-x} - 40 = 0$$

$$\Rightarrow 9e^{2x} - 20e^x + 4 = 0 \Rightarrow \left(9e^x - 2\right)\left(e^x - 2\right) = 0$$

$$\Rightarrow e^x = \frac{2}{9} \text{ or } e^x = 2 \quad \Rightarrow x = \ln\left(\frac{2}{9}\right) \text{or } x = \ln 2$$

REVISION SHEET – FP2 (AQA)

SERIES AND INDUCTION

The main ideas are:

- Summing Series using standard formulae
- Telescoping
- Proof by Induction

Before the exam you should know:

- The standard formula:

$$\sum_{r=1}^{n} r, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r^3$$

- And be able to spot that a series like

$$(1\times 2)+(2\times 3)+...+n(n+1)$$

can be written in sigma notation as:

$$\sum_{r=1}^{n} r(r+1)$$

- How to do proof by induction

Summing Series

Using standard formulae

Fluency is required in manipulating and simplify standard formulae sums like:

$$\sum_{r=1}^{n} r\left(r^2+1\right)=\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r=\frac{n^2\left(n+1\right)^2}{4}+\frac{n\left(n+1\right)}{2}$$

$$=\frac{1}{4}n\left(n+1\right)\left[n\left(n+1\right)+2\right]$$

$$=\frac{1}{4}n\left(n+1\right)\left(n^2+n+2\right).$$

The Method of Differences (Telescoping)

Since $\frac{r+4}{r(r+1)(r+2)}=\frac{2}{r}-\frac{3}{r+1}+\frac{1}{r+2}$ (frequently in exam questions you are told to show that this is true first) it is possible to demonstrate that:

$$\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}=\left(2-\frac{3}{2}+\frac{1}{3}\right)+\left(\frac{2}{2}-\frac{3}{3}+\frac{1}{4}\right)+\left(\frac{2}{3}-\frac{3}{4}+\frac{1}{5}\right)+...$$

$$+\left(\frac{2}{n-2}-\frac{3}{n-1}+\frac{1}{n}\right)+\left(\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}\right)+\left(\frac{2}{n}-\frac{3}{n+1}+\frac{1}{n+2}\right)$$

In this kind of expression many terms cancel with each other. For example, the (+)$\frac{1}{3}$ in the first bracket cancels with the (–)$\frac{3}{3}$ in the second bracket and the (+)$\frac{2}{3}$ in the third bracket. (subsequent fractions that are cancelling are doing so with terms in the "…" part of the sum.)

This leaves $\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}=\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$.

Proof by Induction

1. Using proof by induction to prove a formula for the summation of a series,

$$\text{E.g., Prove that } \sum_{r=1}^{n}(2r-1)=n^2 .$$

2. Other miscellaneous questions. These are usually very easy, in fact easier than the questions which fall into the categories above, so long as you don't panic, keep a clear head and apply what you know.

E.g., show that if $M=\begin{pmatrix}5 & 8\\ -2 & -3\end{pmatrix}$ then $M^n=\begin{pmatrix}1+4n & 8n\\ -2n & 1-4n\end{pmatrix}$ for all natural numbers *n*.

Example

Prove by induction that, for all positive integers *n*, $\sum_{r=1}^{n}3r+1=\frac{1}{2}n(3n+5)$.

Solution

When $n = 1$ the left hand side equals $(3\times1)+1=4$. The right hand side is $\frac{1}{2}\times1\times((3\times1)+5)=4$. So the statement is true when $n = 1$.

Assume the statement is true when $n = k$. In other words $\sum_{r=1}^{k}3r+1=\frac{1}{2}k(3k+5)$.

It must now be shown that the statement would be true when $n = k + 1$, i.e. that $\sum_{r=1}^{k+1}3r+1=\frac{1}{2}(k+1)(3k+8)$.

Now,

$$\begin{aligned}\sum_{r=1}^{k+1}(3r+1)&=\sum_{r=1}^{k}(3r+1)+(3(k+1)+1)\\&=\frac{1}{2}k(3k+5)+(3k+4)\\&=\frac{1}{2}\left[3k^2+5k+6k+8\right]\\&=\frac{1}{2}\left[3k^2+11k+8\right]\\&=\frac{1}{2}(k+1)(3k+8)\end{aligned}$$

So the statement is true when $n = 1$ and if it's true when $n = \text{k}$, then it's also true when $n = k + 1$.

Hence, by induction the statement is true for all positive integers, *n*.