Pure
Further Mathematics 2

Revision Notes

October 2016

Further Pure 2

1 Inequalities

Algebraic solutions

Remember that if you multiply both sides of an inequality by a negative number, you must turn the inequality sign round: $2x > 3 \Rightarrow -2x < -3$.

A difficulty occurs when multiplying both sides by, for example, $(x - 2)$; this expression is sometimes positive ($x > 2$), sometimes negative ($x < 2$) and sometimes zero ($x = 2$). In this case we multiply both sides by $(x - 2)^2$, which is always positive (provided that $x \neq 2$).

Example 1: Solve the inequality $2x + 3 < \frac{x^2}{x-2}$, $x \neq 2$

Solution: Multiply both sides by $(x - 2)^2$ — we can do this since $(x - 2) \neq 0$

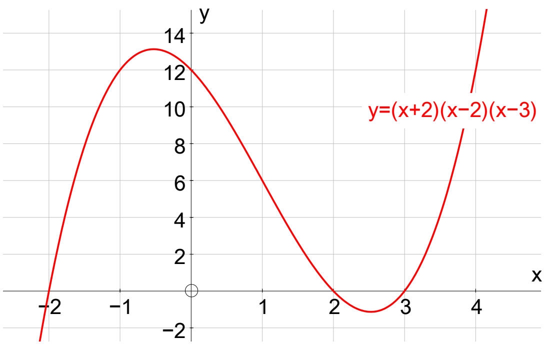
$\Rightarrow \quad (2x+3)(x-2)^2 < x^2(x-2)$ — DO **NOT** MULTIPLY OUT

$\Rightarrow \quad (2x+3)(x-2)^2 - x^2(x-2) < 0$

$\Rightarrow \quad (x-2)(2x^2 - x - 6 - x^2) < 0$

$\Rightarrow \quad (x-2)(x-3)(x+2) < 0$

$\Rightarrow \quad x < -2$, or $2 < x < 3$, below x-axis



Note – care is needed when the inequality is $\leq$ or $\geq$.

Example 2: Solve the inequality $\frac{x}{x+1} \geq \frac{2}{x+3}$, $x \neq -1$, $x \neq -3$

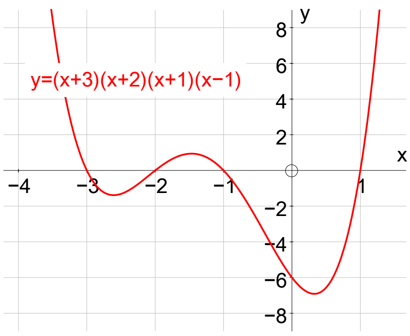
Solution: Multiply both sides by $(x + 1)^2(x + 3)^2$ — which cannot be zero

$\Rightarrow \quad x(x+1)(x+3)^2 \geq 2(x+3)(x+1)^2$ — DO **NOT** MULTIPLY OUT

$\Rightarrow \quad x(x+1)(x+3)^2 - 2(x+3)(x+1)^2 \geq 0$

$\Rightarrow \quad (x+1)(x+3)(x^2 + 3x - 2x - 2) \geq 0$

$\Rightarrow \quad (x+1)(x+3)(x+2)(x-1) \geq 0$



from sketch it looks as though the solution is

$$x \leq -3 \quad \text{or} \quad -2 \leq x \leq -1 \quad \text{or} \quad x \geq 1$$

BUT since $x \neq -1$, $x \neq -3$,

the solution is $x < -3$ or $-2 \leq x < -1$ or $x \geq 1$, above the x-axis

Graphical solutions

Example 1: On the same diagram sketch the graphs of $y = \frac{2x}{x+3}$ and $y = x - 2$.

Use your sketch to solve the inequality $\frac{2x}{x+3} \geq x - 2$

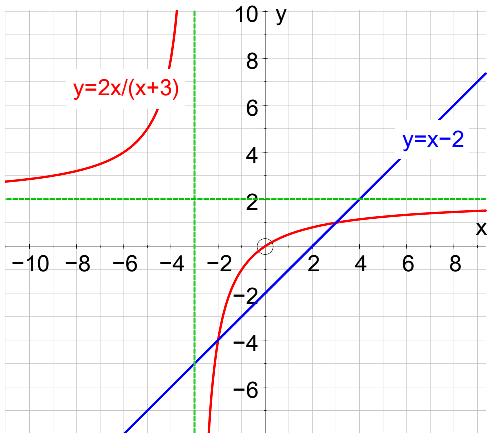
Solution: First find the points of intersection of the two graphs

$\Rightarrow \quad \frac{2x}{x+3} = x - 2$

$\Rightarrow \quad 2x = x^2 + x - 6$

$\Rightarrow \quad 0 = (x-3)(x+2)$

$\Rightarrow \quad x = -2 \text{ or } 3$



From the sketch we see that

$x < -3$ or $-2 \leq x \leq 3$. Note that $x \neq -3$

For inequalities involving $|2x-5|$ etc., it is often essential to sketch the graphs first.

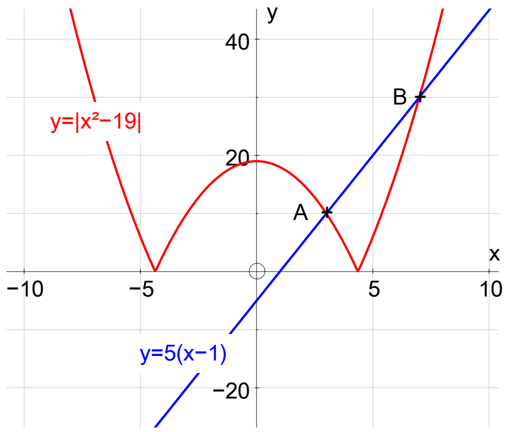
Example 2: Solve the inequality $|x^2 - 19| < 5(x - 1)$.

Solution: It is essential to sketch the curves first in order to see which solutions are needed.

To find the point A, we need to solve

$-(x^2 - 19) = 5x - 5 \quad \Rightarrow \quad x^2 + 5x - 24 = 0$

$\Rightarrow \quad (x+8)(x-3) = 0 \quad \Rightarrow \quad x = -8 \text{ or } 3$



From the sketch $x \neq -8 \quad \Rightarrow \quad x = 3$

To find the point B, we need to solve

$+(x^2 - 19) = 5x - 5 \quad \Rightarrow \quad x^2 - 5x - 14 = 0$

$\Rightarrow \quad (x-7)(x+2) = 0 \quad \Rightarrow \quad x = -2 \text{ or } 7$

From the sketch $x \neq -2 \quad \Rightarrow \quad x = 7$

$\Rightarrow$ the solution of $|x^2 - 19| < 5(x - 1)$ is $3 < x < 7$

2 Series – Method of Differences

The trick here is to write each line out in full and see what cancels when you add.

Do not be tempted to work each term out – you will lose the pattern which lets you cancel when adding.

Example 1: Write $\frac{1}{r(r+1)}$ in partial fractions, and then use the method of differences to find the sum $\sum_{r=1}^{n} \frac{1}{\mathrm{r(r+1)}} = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n(n+1)}$.

Solution: $\frac{1}{r(r+1)} = \frac{1}{r} - \frac{1}{r+1}$

put $r = 1 \Rightarrow \frac{1}{1\times 2} = \frac{1}{1} - \frac{1}{2}$

put $r = 2 \Rightarrow \frac{1}{2\times 3} = \frac{1}{2} - \frac{1}{3}$

put $r = 3 \Rightarrow \frac{1}{3\times 4} = \frac{1}{3} - \frac{1}{4}$

etc.

put $r = n \Rightarrow \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

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adding $\Rightarrow \sum_{1}^{n} \frac{1}{r(r+1)} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

Example 2: Write $\frac{2}{r(r+1)(r+2)}$ in partial fractions, and then use the method of differences to find the sum $\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{1\times2\times3} + \frac{1}{2\times3\times4} + \frac{1}{3\times4\times5} + \cdots + \frac{1}{n(n+1)(n+2)}$.

Solution: $\frac{2}{r(r+1)(r+2)} = \frac{1}{r} - \frac{2}{r+1} + \frac{1}{r+2}$

put $r = 1$ $\Rightarrow$ $\frac{2}{1\times2\times3} = \frac{1}{1} - \frac{2}{2} + \frac{1}{3}$

put $r = 2$ $\Rightarrow$ $\frac{2}{2\times3\times4} = \frac{1}{2} - \frac{2}{3} + \frac{1}{4}$

put $r = 3$ $\Rightarrow$ $\frac{2}{3\times4\times5} = \frac{1}{3} - \frac{2}{4} + \frac{1}{5}$

put $r = 4$ $\Rightarrow$ $\frac{2}{4\times5\times6} = \frac{1}{4} - \frac{2}{5} + \frac{1}{6}$

⋮

etc.

⋮

put $r = n-1$ $\Rightarrow$ $\frac{2}{(n-1)n(n+1)} = \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n+1}$

put $r = n$ $\Rightarrow$ $\frac{2}{n(n+1)(n+2)} = \frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}$

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adding $\Rightarrow$ $\sum_{1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{1}{1} - \frac{2}{2} + \frac{1}{2} + \frac{1}{n+1} - \frac{2}{n+1} + \frac{1}{n+2}$

$$= \frac{1}{2} - \frac{1}{n+1} + \frac{1}{n+2}$$

$$= \frac{n^2+3n+2-2n-4+2n+2}{2(n+1)(n+2)}$$

$\Rightarrow$ $$\sum_{1}^{n} \frac{2}{r(r+1)(r+2)} = \frac{n^2+3n}{2(n+1)(n+2)}$$

$\Rightarrow$ $$\sum_{1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{n^2+3n}{4(n+1)(n+2)}$$

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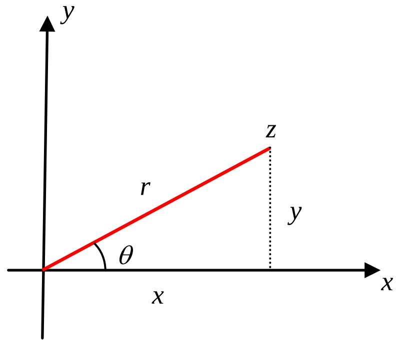
3 Complex Numbers

Modulus and Argument



The modulus of $z = x + iy$ is the length of z

$\Rightarrow \quad r = |z| = \sqrt{x^2 + y^2}$

and the argument of z is the angle made by z with the positive x-axis, $-\pi < \arg z \leqslant \pi$.

N.B. $\arg z$ is **not always** equal to $\tan^{-1}\left(\frac{y}{x}\right)$

Properties

$$z = r\cos\theta + i\, r\sin\theta$$

$$|zw| = |z|\,|w|, \quad \text{and} \quad \left|\frac{z}{w}\right| = \frac{|z|}{|w|}$$

$$\arg(zw) = \arg z + \arg w, \quad \text{and} \quad \arg\left(\frac{z}{w}\right) = \arg z - \arg w$$

Euler's Relation $e^{i\theta}$

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$$

Example: Express $5e^{\left(\frac{i3\pi}{4}\right)}$ in the form $x + iy$.

Solution: $5e^{\left(\frac{i3\pi}{4}\right)} = 5\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

$$= \frac{-5\sqrt{2}}{2} + i\frac{5\sqrt{2}}{2}$$

Multiplying and dividing in mod-arg form

$$re^{i\theta} \times se^{i\phi} = rs\, e^{i(\theta+\phi)}$$

$$\equiv \quad (r\cos\theta + i\, r\sin\theta) \times (s\cos\phi + i\, s\sin\phi) = rs\cos(\theta+\phi) + i\, rs\sin(\theta+\phi)$$

and

$$re^{i\theta} \div se^{i\phi} = \frac{r}{s}\, e^{i(\theta-\phi)}$$

$$\equiv (r\cos\theta + i\, r\sin\theta) \div (s\cos\phi + i\, s\sin\phi) = \frac{r}{s}\cos(\theta-\phi) + i\,\frac{r}{s}\sin(\theta-\phi)$$

De Moivre's Theorem

$$\left(re^{i\theta}\right)^n = r^n e^{in\theta} \equiv (r\cos\theta + i\, r\sin\theta)^n = (r^n\cos n\theta + i\, r^n \sin n\theta)$$

Applications of De Moivre's Theorem

Example: Express $\sin 5\theta$ in terms of $\sin\theta$ only.

Solution: From De Moivre's Theorem we know that

$$\cos 5\theta + i\sin 5\theta = (\cos\theta + i\sin\theta)^5$$

$$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$$

Equating imaginary parts

$$\Rightarrow \quad \sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

$$= 5(1-\sin^2\theta)^2\sin\theta - 10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta$$

$$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

$z^n + \frac{1}{z^n} = 2\cos n\theta$ and $z^n - \frac{1}{z^n} = 2i\sin n\theta$

$$z = \cos\theta + i\sin\theta$$

$$\Rightarrow \quad z^n = (\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

and $$\frac{1}{z^n} = z^{-n} = (\cos\theta + i\sin\theta)^{-n} = (\cos n\theta - i\sin n\theta)$$

from which we can show that

$$\left(z + \frac{1}{z}\right) = 2\cos\theta \quad \text{and} \quad \left(z - \frac{1}{z}\right) = 2i\sin\theta$$

$$z^n + \frac{1}{z^n} = 2\cos n\theta \quad \text{and} \quad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Example: Express $\sin^5\theta$ in terms of $\sin 5\theta$, $\sin 3\theta$ and $\sin\theta$.

Solution: Here we are dealing with $\sin\theta$, so we use

$$(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5$$

$$\Rightarrow \quad 32i^5\sin^5\theta = z^5 - 5z^4\left(\frac{1}{z}\right) + 10z^3\left(\frac{1}{z^2}\right) - 10z^2\left(\frac{1}{z^3}\right) + 5z\left(\frac{1}{z^4}\right) - \left(\frac{1}{z^5}\right)$$

$$\Rightarrow \quad 32i\sin^5\theta = \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$\Rightarrow \quad 32i\sin^5\theta = 2i\sin 5\theta - 5\times 2i\sin 3\theta + 10\times 2i\sin\theta$$

$$\Rightarrow \quad \sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

n^{th} roots of a complex number

The technique is the same for finding n^{th} roots of any complex number.

Example: Find the 4^{th} roots of $8\sqrt{2} + 8\sqrt{2}i$, and show the roots on an Argand Diagram.

Solution: We need to solve the equation $z^4 = 8\sqrt{2} + 8\sqrt{2}i$

1. Let $z = r\cos\theta + i\, r\sin\theta$

   $\Rightarrow \quad z^4 = r^4(\cos 4\theta + i\sin 4\theta)$

2. $|\, 8\sqrt{2} + 8\sqrt{2}i\,| = 8\sqrt{2+2} = 16$ and $\arg(8\sqrt{2} + 8\sqrt{2}i) = \frac{\pi}{4}$

   $\Rightarrow \quad 8\sqrt{2} + 8\sqrt{2}i = 16\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

3. Then $z^4 = 8\sqrt{2} + 8\sqrt{2}i$

   becomes $r^4(\cos 4\theta + i\sin 4\theta) = 16\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

   $= 16\left(\cos\frac{9\pi}{4} + i\sin\frac{9\pi}{4}\right)$ adding 2π

   $= 16\left(\cos\frac{17\pi}{4} + i\sin\frac{17\pi}{4}\right)$ adding 2π

   $= 16\left(\cos\frac{25\pi}{4} + i\sin\frac{25\pi}{4}\right)$ adding 2π

4. $\Rightarrow \quad r^4 = 16$ and $4\theta = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \frac{25\pi}{4}$

   $\Rightarrow \quad r = 2$ and $\theta = \frac{\pi}{16}, \frac{9\pi}{16}, \frac{17\pi}{16} = \frac{-15\pi}{16}, \frac{25\pi}{16} = \frac{-7\pi}{16}$; $\quad -\pi < \arg z \leqslant \pi$

5. $\Rightarrow$ roots are

   $z_1 = 2\left(\cos\frac{\pi}{16} + i\sin\frac{\pi}{16}\right) = 1{\cdot}962 + 0{\cdot}390\, i$

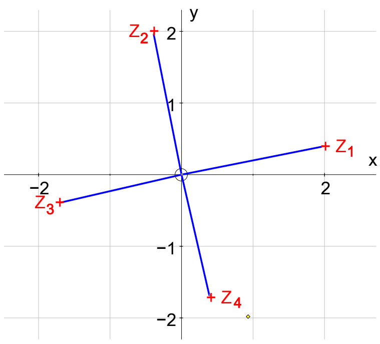
   $z_2 = 2\left(\cos\frac{9\pi}{16} + i\sin\frac{9\pi}{16}\right) = -0{\cdot}390 + 1{\cdot}962\, i$

   $z_3 = 2\left(\cos\frac{-15\pi}{16} + i\sin\frac{-15\pi}{16}\right) = -1{\cdot}962 - 0{\cdot}390\, i$

   $z_4 = 2\left(\cos\frac{-7\pi}{16} + i\sin\frac{-7\pi}{16}\right) = 0{\cdot}390 - 1{\cdot}962\, i$

Notice that the roots are symmetrically placed around the origin, and the angle between roots is $\frac{2\pi}{4} = \frac{\pi}{2}$

The angle between the n^{th} roots will always be $\frac{2\pi}{n}$.



For sixth roots the angle between roots will be $\frac{2\pi}{6} = \frac{\pi}{3}$, and so on.

Roots of polynomial equations with real coefficients

1. **Any** polynomial equation with real coefficients,
   $a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots\ldots a_2x^2 + a_1x + a_0 = 0$,   …….. (I)
   where all a_i are real, has a complex solution
2. $\Rightarrow$ **any** complex n^{th} degree polynomial can be factorised into n linear factors over the complex numbers
3. If $z = a + ib$ is a root of (I), then its conjugate, $a - ib$ is also a root – see FP1.
4. By pairing factors with conjugate pairs we can say that any polynomial with real coefficients can be factorised into a combination of linear and quadratic factors over the real numbers.

Example: Given that $3 - 2i$ is a root of $z^3 - 5z^2 + 7z + 13 = 0$

(*a*) Factorise over the real numbers

(*b*) Find all three real roots

Solution:

(*a*) $3 - 2i$ is a root $\Rightarrow$ $3 + 2i$ is also a root

$\Rightarrow$ $(z - (3 - 2i))(z - (3 + 2i)) = (z^2 - 6z + 13)$ is a factor

$\Rightarrow$ $z^3 - 5z^2 + 7z + 13 = (z^2 - 6z + 13)(z + 1)$ by inspection

(*b*) $\Rightarrow$ roots are $z = 3 - 2i,\ 3 + 2i$ and -1

Loci on an Argand Diagram

Two basic ideas

1. $|z - w|$ is the distance from w to z .
2. arg $(z - (1 + i))$ is the angle made by the *half* line joining $(1+i)$ to z, with the x-axis.

Example 1:

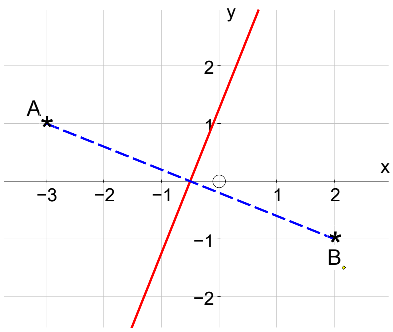
$|z - 2 - i| = 3$ is a circle with centre $(2 + i)$ and radius 3

Example 2:

$|z + 3 - i| = |z - 2 + i|$

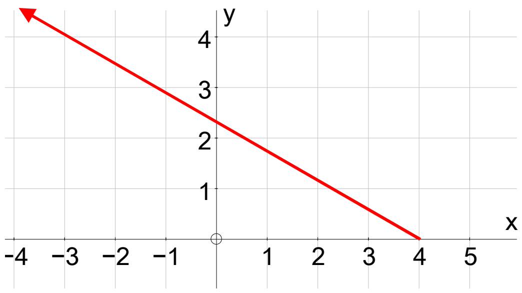
$\Leftrightarrow$ $|z - (-3 + i)| = |z - (2 - i)|$

is the locus of all points which are equidistant from the points
$A\,(-3, 1)$ and $B\,(2, -1)$, and so is the perpendicular bisector of AB.



Example 3:

arg $(z - 4) = \frac{5\pi}{6}$ is a half line, from (4, 0), making an angle of $\frac{5\pi}{6}$ with the x-axis.



Example 4:

$|z - 3| = 2|z + 2i|$ is a circle (Apollonius's circle).

To find its equation, put $z = x + iy$

$\Rightarrow \quad |(x - 3) + iy| = 2|x + i(y + 2)|$ square both sides

$\Rightarrow \quad (x - 3)^2 + y^2 = 4(x^2 + (y + 2)^2)$ leading to

$\Rightarrow \quad 3x^2 + 6x + 3y^2 + 16y + 7 = 0$

$\Rightarrow \quad (x + 1)^2 + \left(y + \frac{8}{3}\right)^2 = \frac{52}{9}$

which is a circle with centre $(-1, \frac{-8}{3})$, and radius $\frac{2\sqrt{13}}{3}$.

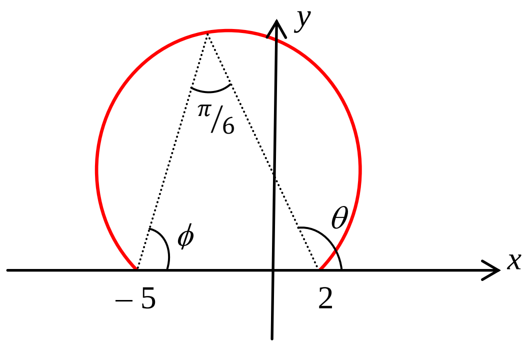
Example 5:

arg $\left(\frac{z-2}{z+5}\right) = \frac{\pi}{6}$

$\Rightarrow \quad \arg(z - 2) - \arg(z + 5) = \frac{\pi}{6}$

$\Rightarrow \quad \theta - \phi = \frac{\pi}{6}$

which gives the arc of the circle as shown.



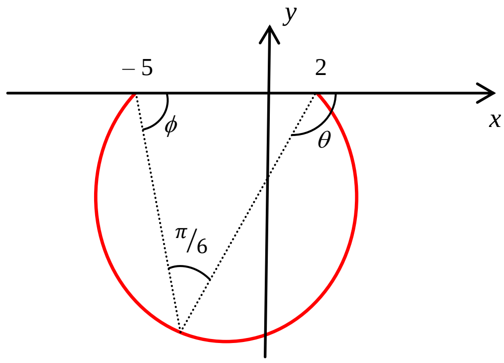
N.B.

The corresponding arc below the x-axis would have equation

$$\arg\left(\frac{z-2}{z+5}\right) = -\frac{\pi}{6}$$

as $\theta - \phi$ would be negative in this picture.

(θ is a 'larger negative number' than ϕ.)



Transformations of the Complex Plane

Always start from the z-plane and transform to the w-plane, $z = x + iy$ and $w = u + iv$.

Example 1: Find the image of the circle $|z - 5| = 3$ under the transformation $w = \frac{1}{z-2}$.

Solution: *First* rearrange to find z

$$w = \frac{1}{z-2} \Rightarrow z - 2 = \frac{1}{w} \qquad \Rightarrow \qquad z = \frac{1}{w} + 2$$

Second substitute in equation of circle

$$\Rightarrow \quad \left|\frac{1}{w} + 2 - 5\right| = 3 \qquad \Rightarrow \qquad \left|\frac{1-3w}{w}\right| = 3$$

$$\Rightarrow \quad |1 - 3w| = 3|w| \qquad \Rightarrow \qquad 3\left|\frac{1}{3} - w\right| = 3|w|$$

$$\Rightarrow \quad \left|w - \frac{1}{3}\right| = |w|$$

which is the equation of the perpendicular bisector of the line joining 0 to $\frac{1}{3}$,

$\Rightarrow$ the image is the line $u = \frac{1}{6}$

Always consider the 'modulus technique' (above) first;

if this does not work then use the *u + iv* method shown below.

Example 2: Show that the image of the line $x + 4y = 4$ under the transformation $w = \frac{1}{z-3}$ is a circle, and find its centre and radius.

Solution: *First* rearrange to find $z \Rightarrow z = \frac{1}{w} + 3$

The 'modulus technique' is not suitable here.

$z = x + iy$ and $w = u + iv$

$$\Rightarrow \quad z = \frac{1}{w} + 3 = \frac{1}{u+iv} + 3 = \frac{1}{u+iv} \times \frac{u-iv}{u-iv} + 3$$

$$\Rightarrow \quad x + iy = \frac{u-iv}{u^2+v^2} + 3$$

Equating real and imaginary parts $x = \frac{u}{u^2+v^2} + 3$ and $y = \frac{-v}{u^2+v^2}$

$$\Rightarrow \quad x + 4y = 4 \quad \text{becomes} \quad \frac{u}{u^2+v^2} + 3 - \frac{4v}{u^2+v^2} = 4$$

$$\Rightarrow \quad u^2 - u + v^2 + 4v = 0$$

$$\Rightarrow \quad \left(u - \frac{1}{2}\right)^2 + (v + 2)^2 = \frac{17}{4}$$

which is a circle with centre $\left(\frac{1}{2}, -2\right)$ and radius $\frac{\sqrt{17}}{2}$.

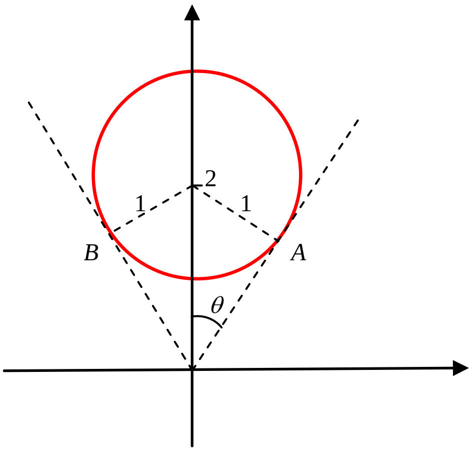
There are many more examples in the book, but these are the two important techniques.

Loci and geometry

It is always important to think of diagrams.

Example: z lies on the circle $|z - 2i| = 1$.
Find the greatest and least values of $\arg z$.

Solution: Draw a picture!



The greatest and least values of $\arg z$ will occur at B and A.

Trigonometry tells us that

$\theta = \frac{\pi}{6}$

and so greatest and least values of

$\arg z$ are $\frac{2\pi}{3}$ and $\frac{\pi}{3}$

4 First Order Differential Equations

Separating the variables, families of curves

Example: Find the general solution of

$$\frac{dy}{dx} = \frac{y}{2x(x+1)}, \quad \text{for } x > 0,$$

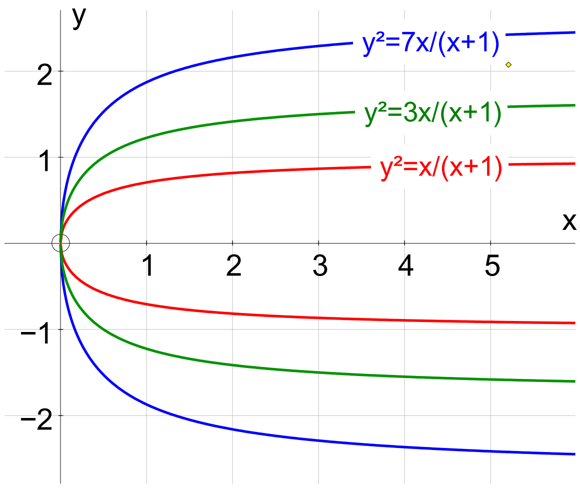
and sketch some of the family of solution curves.

Solution: $\frac{dy}{dx} = \frac{y}{2x(x+1)} \Rightarrow \int \frac{2}{y}\,dy = \int \frac{1}{x(x+1)}\,dx = \int \frac{1}{x} - \frac{1}{x+1}\,dx$

$$\Rightarrow \quad 2\ln y = \ln x - \ln(x+1) + \ln A$$

$$\Rightarrow \quad y^2 = \frac{Ax}{x+1}$$

Thus for varying values of A and for $x > 0$, we have



Exact Equations

In an exact equation the L.H.S. is an exact derivative (really a preparation for Integrating Factors).

Example: Solve $\sin x \frac{dy}{dx} + y\cos x = 3x^2$

Solution: Notice that the L.H.S. is an exact derivative

$$\sin x \frac{dy}{dx} + y\cos x = \frac{d}{dx}(y\sin x)$$

$$\Rightarrow \quad \frac{d}{dx}(y\sin x) = 3x^2$$

$$\Rightarrow \quad y\sin x = \int 3x^2\,dx = x^3 + c$$

$$\Rightarrow \quad y = \frac{x^3 + c}{\sin x}$$

Integrating Factors

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only.

In this case, multiply both sides by an Integrating Factor, $R = e^{\int P\,dx}$.

The L.H.S. will now be an exact derivative, $\frac{d}{dx}(Ry)$.

Proceed as in the above example.

Example: Solve $x\frac{dy}{dx} + 2y = 1$

Solution: First divide through by x

$\Rightarrow \quad \frac{dy}{dx} + \frac{2}{x}y = \frac{1}{x}$ now in the correct form

Integrating Factor, I.F., is $R = e^{\int P\,dx} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = x^2$

$\Rightarrow \quad x^2\frac{dy}{dx} + 2xy = x$ multiplying by x^2

$\Rightarrow \quad \frac{d}{dx}(x^2y) = x,$ check that it is an exact derivative

$\Rightarrow \quad x^2 y = \int x\,dx = \frac{x^2}{2} + c$

$\Rightarrow \quad y = \frac{1}{2} + \frac{c}{x^2}$

Using substitutions

Example 1*:* Use the substitution $y = vx$ (where v is a function of x) to solve the equation $\frac{dy}{dx} = \frac{3yx^2 + y^3}{x^3 + xy^2}$.

Solution: $y = vx \quad \Rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{3yx^2+y^3}{x^3+xy^2} \quad \Rightarrow \quad v + x\frac{dv}{dx} = \frac{3(vx)x^2 + (vx)^3}{x^3 + x(vx)^2} = \frac{3v + v^3}{1+v^2}$

and we can now separate the variables

$\Rightarrow \quad x\frac{dv}{dx} = \frac{3v+v^3}{1+v^2} - v = \frac{3v + v^3 - v - v^3}{1+v^2} = \frac{2v}{1+v^2}$

$\Rightarrow \quad \frac{1+v^2}{2v}\frac{dv}{dx} = \frac{1}{x}$

$\Rightarrow \quad \int \frac{1}{2v} + \frac{v}{2}\,dv = \int \frac{1}{x}\,dx$

$\Rightarrow \quad \frac{1}{2}\ln v + \frac{v^2}{4} = \ln x + c$

But $v = \frac{y}{x}$, $\quad \Rightarrow \quad \frac{1}{2}\ln\frac{y}{x} + \frac{y^2}{4x^2} = \ln x + c$

$\Rightarrow \quad 2x^2 \ln y + y^2 = 6x^2 \ln x + c'x^2$ c' is new arbitrary constant

and I would not like to find y!!!

If told to use the substitution $v = \frac{y}{x}$, rewrite as $y = vx$ and proceed as in the above example.

Example 2: Use the substitution $y = \frac{1}{z}$ to solve the differential equation $\frac{dy}{dx} = y^2 + y\cot x$.

Solution: $y = \frac{1}{z} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-1}{z^2}\frac{dz}{dx}$

$\Rightarrow \quad \frac{-1}{z^2}\frac{dz}{dx} = \frac{1}{z^2} + \frac{1}{z}\cot x$

$\Rightarrow \quad \frac{dz}{dx} + z\cot x = -1$

Integrating factor is $R = e^{\int \cot x \; dx} = e^{\ln(\sin x)} = \sin x$

$\Rightarrow \quad \sin x\frac{dz}{dx} + z\cos x = -\sin x$

$\Rightarrow \quad \frac{d}{dx}(z\sin x) = -\sin x$ check that it is an exact derivative

$\Rightarrow \quad z\sin x = \cos x + c$

$\Rightarrow \quad z = \frac{\cos x + c}{\sin x}$ but $z = \frac{1}{y}$

$\Rightarrow \quad y = \frac{\sin x}{\cos x + c}$

Example 3: Use the substitution $z = x + y$ to solve the differential equation $\frac{dy}{dx} = \cos(x + y)$

Solution: $z = x + y \quad \Rightarrow \quad \frac{dz}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \quad \frac{dz}{dx} = 1 + \cos z$

$\Rightarrow \quad \int \frac{1}{1+\cos z}\, dz = \int dx$ separating the variables

$\Rightarrow \quad \int \frac{1}{2}\sec^2\left(\frac{z}{2}\right)\, dz = x + c$ $1 + \cos z = 1 + 2\cos^2\left(\frac{z}{2}\right) - 1 = 2\cos^2\left(\frac{z}{2}\right)$

$\Rightarrow \quad \tan\left(\frac{z}{2}\right) = x + c$

But $z = x + y \quad \Rightarrow \quad \tan\left(\frac{x+y}{2}\right) = x + c$

5 Second Order Differential Equations

Linear with constant coefficients

$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ where a, b and c are constants.

(1) when *f* (*x*) = 0

First write down the Auxiliary Equation, A.E

A.E. $am^2 + bm + c = 0$

and solve to find the roots $m = \alpha$ or β

(i) If α and β are both real numbers, and if $\alpha \neq \beta$
then the Complimentary Function, C.F., is
$y = A\,e^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants of integration

(ii) If α and β are both real numbers, and if $\alpha = \beta$
then the Complimentary Function, C.F., is
$y = (A + Bx)\,e^{\alpha x}$, where A and B are arbitrary constants of integration

(iii) If α and β are both complex numbers, and if $\alpha = a + ib,\ \beta = a - ib$
then the Complimentary Function, C.F.,
$y = e^{ax}(A\sin bx + B\cos bx)$,
where A and B are arbitrary constants of integration

Example 1: Solve $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$

Solution: A.E. is $m^2 + 2m - 3 = 0$

$\Rightarrow$ $(m-1)(m+3) = 0$

$\Rightarrow$ $m = 1$ or -3

$\Rightarrow$ $y = Ae^x + Be^{-3x}$ when $f(x) = 0$, the C.F. is the solution

Example 2: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Solution: A.E. is $m^2 + 6m + 9 = 0$

$\Rightarrow$ $(m+3)^2 = 0$

$\Rightarrow$ $m = -3$ (and -3) repeated root

$\Rightarrow$ $y = (A + Bx)e^{-3x}$ when $f(x) = 0$, the C.F. is the solution

Example 3: Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$

Solution: A.E. is $m^2 + 4m + 13 = 0$

$\Rightarrow$ $(m + 2)^2 = -9 = (3i)^2$

$\Rightarrow$ $(m + 2) = \pm 3i$

$\Rightarrow$ $m = -2 - 3i$ or $-2 + 3i$

$\Rightarrow$ $y = e^{-2x}(A \sin 3x + B \cos 3x)$ when $f(x) = 0$, the C.F. is the solution

(2) when $f(x) \neq 0$, Particular Integrals

First proceed as in (1) to find the Complimentary Function, then use the rules below to find a Particular Integral, P.I.

Second the General Solution, G.S. , is found by adding the C.F. and the P.I.

$\Rightarrow$ **G.S. = C.F. + P.I.**

Note that it does not matter what P.I. you use, so you might as well find the easiest, which is what these rules do.

(1) $f(x) = e^{kx}$.

Try $y = Ae^{kx}$
unless e^{kx} appears, on its own, in the C.F., in which case try $y = Cxe^{kx}$
unless xe^{kx} appears, on its own, in the C.F., in which case try $y = Cx^2e^{kx}$.

(2) $f(x) = \sin kx$ **or** $f(x) = \cos kx$

Try $y = C \sin kx + D \cos kx$
unless $\sin kx$ or $\cos kx$ appear in the C.F., on their own, in which case
try $y = x(C \sin kx + D \cos kx)$

(3) $f(x)$ **= a polynomial of degree** n**.**

Try $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$
unless a number, on its own, appears in the C.F., in which case
try $f(x) = x(a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0)$
i.e. try $f(x)$ = a polynomial of degree n.

(4) In general
to find a P.I., try something like $f(x)$, unless this appears in the C.F. (or if there is a problem), then try something like $x f(x)$.

Example 1: Solve $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 5y = 2x$

Solution: A.E. is $m^2 + 6m + 5 = 0$

$\Rightarrow$ $(m + 5)(m + 1) = 0$ $\Rightarrow$ $m = -5$ or -1

$\Rightarrow$ C.F. is $y = Ae^{-5x} + Be^{-x}$

For the P.I., try $y = Cx + D$

$\Rightarrow$ $\frac{dy}{dx} = C$ and $\frac{d^2y}{dx^2} = 0$

Substituting in the differential equation gives

$0 + 6C + 5(Cx + D) = 2x$

$\Rightarrow$ $5C = 2$ comparing coefficients of x

$\Rightarrow$ $C = \frac{2}{5}$

and $6C + 5D = 0$ comparing constant terms

$\Rightarrow$ $D = \frac{-12}{25}$

$\Rightarrow$ P.I. is $y = \frac{2}{5}x - \frac{12}{25}$

$\Rightarrow$ G.S. is $y = Ae^{-5x} + Be^{-x} + \frac{2}{5}x - \frac{12}{25}$

Example 2: Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = e^{3x}$

Solution: A.E. is is $m^2 - 6m + 9 = 0$

$\Rightarrow$ $(m - 3)^2 = 0$

$\Rightarrow$ $m = 3$ repeated root

$\Rightarrow$ C.F. is $y = (Ax + B)e^{3x}$

In this case, both e^{3x} and xe^{3x} appear, on their own, in the C.F.,
so for a P.I. we try $y = Cx^2e^{3x}$

$\Rightarrow$ $\frac{dy}{dx} = 2Cxe^{3x} + 3Cx^2e^{3x}$

and $\frac{d^2y}{dx^2} = 2Ce^{3x} + 6Cxe^{3x} + 6Cxe^{3x} + 9Cx^2e^{3x}$

Substituting in the differential equation gives

$$2Ce^{3x} + 12Cxe^{3x} + 9Cx^2e^{3x} - 6(2Cxe^{3x} + 3Cx^2e^{3x}) + 9\,Cx^2e^{3x} = e^{3x}$$

$\Rightarrow$ $2Ce^{3x} = e^{3x}$

$\Rightarrow$ $C = \frac{1}{2}$

$\Rightarrow$ P.I. is $y = \frac{1}{2}x^2e^{3x}$

$\Rightarrow$ G.S. is $y = (Ax + B)e^{3x} + \frac{1}{2}x^2e^{3x}$

Example 3: Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 4\cos 2t$
given that $x = 0$ and $\dot{x} = 1$ when $t = 0$.

Solution: A.E. is $m^2 - 3m + 2 = 0$

$\Rightarrow$ $m = 1$ or 2

$\Rightarrow$ C.F. is $x = Ae^t + Be^{2t}$

For the P.I. try $x = C\sin 2t + D\cos 2t$ **BOTH sin 2*t* AND cos 2*t* are needed**

$\Rightarrow$ $\dot{x} = 2C\cos 2t - 2D\sin 2t$

and $\ddot{x} = -4C\sin 2t - 4D\cos 2t$

Substituting in the differential equation gives

$(-4C\sin 2t - 4D\cos 2t) - 3(2C\cos 2t - 2D\sin 2t) + 2(C\sin 2t + D\cos 2t) = 4\cos 2t$

$\Rightarrow$ $-2C + 6D = 0$ $\Rightarrow$ $-C + 3D = 0$ comparing coefficients of sin 2*t*

and $-6C - 2D = 4$ $\Rightarrow$ $3C + D = -2$ comparing coefficients of cos 2*t*

$\Rightarrow$ $C = \frac{-3}{5}$ and $D = \frac{-1}{5}$

$\Rightarrow$ P.I. is $x = -\frac{3}{5}\sin 2t - \frac{1}{5}\cos 2t$

$\Rightarrow$ G.S. is $x = Ae^t + Be^{2t} - \frac{3}{5}\sin 2t - \frac{1}{5}\cos 2t$

$\Rightarrow$ $\dot{x} = Ae^t + 2Be^{2t} - \frac{6}{5}\cos 2t + \frac{2}{5}\sin 2t$

$x = 0$ and when $t = 0$ $\Rightarrow$ $0 = A + B - \frac{1}{5}$

and $\dot{x} = 1$ when $t = 0$ $\Rightarrow$ $1 = A + 2B - \frac{6}{5}$

$\Rightarrow$ $A = \frac{-9}{5}$ and $B = 2$

$\Rightarrow$ solution is $x = \frac{-9}{5}e^t + 2e^{2t} - \frac{6}{5}\sin 2t - \frac{2}{5}\cos 2t$

D.E.s of the form $ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x)$

Substitute $x = e^u$

$\Rightarrow \quad \frac{dx}{du} = e^u = x \quad \Rightarrow \quad \frac{du}{dx} = \frac{1}{x}$

and $\quad \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{x}\frac{dy}{du} \quad \Leftrightarrow \quad x\frac{dy}{dx} = \frac{dy}{du}$ **I**

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{d\left(\frac{dy}{du}\right)}{dx}$

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x}\frac{d\left(\frac{dy}{du}\right)}{du}\frac{du}{dx}$ chain rule

$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{du} + \frac{1}{x^2}\frac{d^2y}{du^2}$

$\Rightarrow \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ **II**

Thus we have $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ and $x\frac{dy}{dx} = \frac{dy}{du}$ from **I** and **II**

substituting these in the original equation leads to a second order D.E. with constant coefficients.

Example: Solve the differential equation $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 3y = -2x^2$.

Solution: Using the substitution $x = e^u$, and proceeding as above

$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{du^2} - \frac{dy}{du}$ and $x\frac{dy}{dx} = \frac{dy}{du}$

$\Rightarrow \quad \frac{d^2y}{du^2} - \frac{dy}{du} - 3\frac{dy}{du} + 3y = -2e^{2u}$

$\Rightarrow \quad \frac{d^2y}{du^2} - 4\frac{dy}{du} + 3y = -2e^{2u}$

$\Rightarrow \quad$ A.E. is $m^2 - 4m + 3 = 0$

$\Rightarrow \quad (m-3)(m-1) = 0 \quad \Rightarrow \quad m = 3$ or 1

$\Rightarrow \quad$ C.F. is $y = Ae^{3u} + Be^u$

For the P.I. try $y = Ce^{2u}$

$\Rightarrow \quad \frac{dy}{du} = 2Ce^{2u}$ and $\frac{d^2y}{du^2} = 4Ce^{2u}$

$\Rightarrow \quad 4Ce^{2u} - 8Ce^{2u} + 3Ce^{2u} = -2e^{2u}$

$\Rightarrow \quad C = 2$

$\Rightarrow \quad$ G.S. is $y = Ae^{3u} + Be^u + 2e^{2u}$

But $x = e^u \quad \Rightarrow \quad$ G.S. is $y = Ax^3 + Bx + 2x^2$

6 Maclaurin and Taylor Series

1) Maclaurin series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

2) Taylor series

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2!}f''(a) + \frac{x^3}{3!}f'''(a) + \cdots + \frac{x^n}{n!}f^n(a) + \cdots$$

3) Taylor series – as a power series in $(x - a)$

replacing x by $(x - a)$ in 2) we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

4) Solving differential equations using Taylor series

(a) If we are given the value of y when $x = 0$, then we use the Maclaurin series with

$f(0) = y_0$ — the value of y when $x = 0$

$f'(0) = \left(\frac{dy}{dx}\right)_0$ — the value of $\frac{dy}{dx}$ when $x = 0$

etc. to give

$$f(x) = y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0 + \cdots + \frac{x^n}{n!}\left(\frac{d^ny}{dx^n}\right)_0 + \cdots$$

(b) If we are given the value of y when $x = a$, then we use the Taylor power series with

$f(a) = y_a$ — the value of y when $x = a$

$f'(a) = \left(\frac{dy}{dx}\right)_a$ — the value of $\frac{dy}{dx}$ when $x = a$

etc. to give

$$y = y_a + (x-a)\left(\frac{dy}{dx}\right)_a + \frac{(x-a)^2}{2!}\left(\frac{d^2y}{dx^2}\right)_a + \frac{(x-a)^3}{3!}\left(\frac{d^3y}{dx^3}\right)_a + \cdots$$

NOTE THAT 4 (a) and 4 (b) are not in the formula book, but can easily be found using the results in 1) and 3).

Standard series

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ converges for all real x

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$ converges for all real x

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2n-2}}{(2n-2)!} + \cdots$ converges for all real x

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$ converges for $-1 < x \leq 1$

$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \cdots$ converges for $-1 < x < 1$

Example 1: Find the Maclaurin series for $f(x) = \tan x$, up to and including the term in x^3

Solution: $f(x) = \tan x \quad \Rightarrow \quad f\ \ (0) = 0$

$\Rightarrow \quad f'(x) = \sec^2 x \quad \Rightarrow \quad f'(0) = 1$

$\Rightarrow \quad f''(x) = 2\sec^2 x \tan x \quad \Rightarrow \quad f''(0) = 0$

$\Rightarrow \quad f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x \quad \Rightarrow \quad f'''(0) = 2$

and $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \ldots$

$\Rightarrow \quad \tan x \cong 0 + x \times 1 + \frac{x^2}{2!} \times 0 + \frac{x^3}{3!} \times 2$ up to the term in x^3

$\Rightarrow \quad \tan x \cong x + \frac{x^3}{3}$

Example 2: Using the Maclaurin series for e^x to find an expansion of e^{x+x^2}, up to and including the term in x^3.

Solution: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$\Rightarrow \quad e^{x+x^2} \cong 1 + (x + x^2) + \frac{(x+x^2)^2}{2!} + \frac{(x+x^2)^3}{3!}$ up to the term in x^3

$\cong 1 + x + x^2 + \frac{x^2+2x^3+\cdots}{2!} + \frac{x^3+\cdots}{3!}$ up to the term in x^3

$\Rightarrow \quad e^{x+x^2} \cong 1 + x + \frac{3}{2}x^2 + \frac{7}{6}x^3$ up to the term in x^3

Example 3: Find a Taylor series for $\cot\left(x+\frac{\pi}{4}\right)$, up to and including the term in x^2.

Solution: $f(x) = \cot x$ and we are looking for

$$f\left(x+\frac{\pi}{4}\right) = f\left(\frac{\pi}{4}\right) + xf'\left(\frac{\pi}{4}\right) + \frac{x^2}{2!}f''\left(\frac{\pi}{4}\right)$$

$f(x) = \cot x \quad \Rightarrow \quad f\left(\frac{\pi}{4}\right) = 1$

$\Rightarrow \quad f'(x) = -\text{cosec}^2\, x \quad \Rightarrow \quad f'\left(\frac{\pi}{4}\right) = -2$

$\Rightarrow \quad f''(x) = 2\text{cosec}^2\, x \cot x \quad \Rightarrow \quad f''\left(\frac{\pi}{4}\right) = 4$

$\Rightarrow \quad \cot\left(x+\frac{\pi}{4}\right) \cong 1 - 2x + \frac{x^2}{2!} \times 4$ up to the term in x^2

$\Rightarrow \quad \cot\left(x+\frac{\pi}{4}\right) \cong 1 - 2x + 2x^2$ up to the term in x^2

Example 4: Use a Taylor series to solve the differential equation,

$$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0 \qquad \text{equation } \mathbf{I}$$

up to and including the term in x^3, given that $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$.

In this case the initial value of x is 0, so we shall use

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^n(0) + \cdots$$

$$\Leftrightarrow \quad y = y_0 + x\left(\frac{dy}{dx}\right)_0 + \frac{x^2}{2!}\left(\frac{d^2y}{dx^2}\right)_0 + \frac{x^3}{3!}\left(\frac{d^3y}{dx^3}\right)_0.$$

We already know that $y_0 = 1$ and $\left(\frac{dy}{dx}\right)_0 = 2$ values when $x = 0$

$\Rightarrow \quad \left(\frac{d^2y}{dx^2}\right)_0 = \left(-\frac{1}{y}\left(\frac{dy}{dx}\right)^2 - 1\right)_0 = -5$ values when $x = 0$

equation **I** $\quad y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0$

Differentiating $\quad \Rightarrow \quad y\frac{d^3y}{dx^3} + \frac{dy}{dx}\times\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\times\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Substituting $y_0 = 1$, $\left(\frac{dy}{dx}\right)_0 = 2$ and $\left(\frac{d^2y}{dx^2}\right)_0 = -5$ values when $x = 0$

$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)_0 + 2 \times ({}^-5) + 2 \times 2 \times ({}^-5) + 2 = 0$

$\Rightarrow \quad \left(\frac{d^3y}{dx^3}\right)_0 = 28$

$\Rightarrow \quad$ solution is $\quad y \cong 1 + 2x + \frac{x^2}{2!} \times ({}^-5) + \frac{x^3}{3!} \times 28$

$\Rightarrow \quad y \cong 1 + 2x - \frac{5}{2}x^2 + \frac{14}{3}x^3$

Series expansions of compound functions

Example: Find a polynomial expansion for

$\frac{\cos 2x}{1-3x}$, up to and including the term in x^3.

Solution: Using the standard series

$\cos 2x = 1 - \frac{(2x)^2}{2!} + \cdots$ up to and including the term in x^3

and $(1-3x)^{-1} = 1 + 3x + \frac{-1\times -2}{2!}(-3x)^2 + \frac{-1\times -2\times -3}{3!}(-3x)^3$

$= 1 + 3x + 9x^2 + 27x^3$ up to and including the term in x^3

$\Rightarrow \frac{\cos 2x}{1-3x} = \left(1 - \frac{(2x)^2}{2!}\right)(1 + 3x + 9x^2 + 27x^3)$

$= 1 + 3x + 9x^2 + 27x^3 - 2x^2 - 6x^3$ up to and including the term in x^3

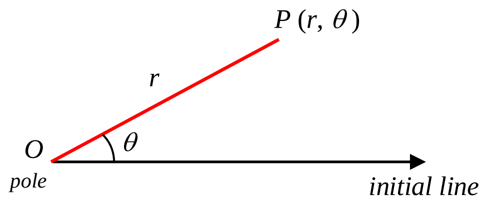
$\Rightarrow \frac{\cos 2x}{1-3x} = 1 + 3x + 7x^2 + 21x^3$ up to and including the term in x^3

7 Polar Coordinates

The polar coordinates of P are $(r,\ \theta)$

$r = OP$, the distance from the origin or *pole*,

and θ is the angle made anti-clockwise with the initial line.



In the Edexcel syllabus r is always taken as positive or 0, and $0 \le \theta < 2\pi$

(But in most books r can be negative, thus $\left(-4,\ \frac{\pi}{2}\right)$ is the same point as $\left(4,\ \frac{3\pi}{2}\right)$)

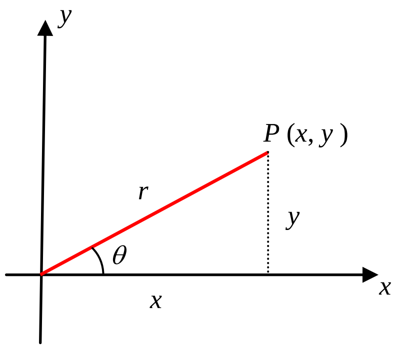
Polar and Cartesian coordinates

From the diagram

$r = \sqrt{x^2 + y^2}$

and $\tan\theta = \frac{y}{x}$ (use sketch to find θ).

$x = r\cos\theta$ and $y = r\sin\theta$.



Sketching curves

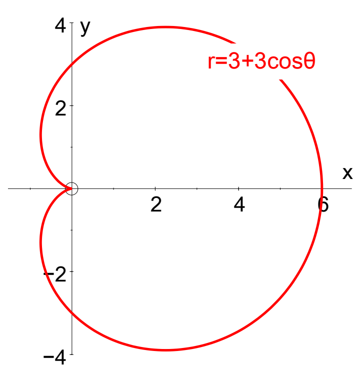
In practice, if you are asked to sketch a curve, it will probably be best to plot a few points. The important values of θ are those for which $r = 0$.

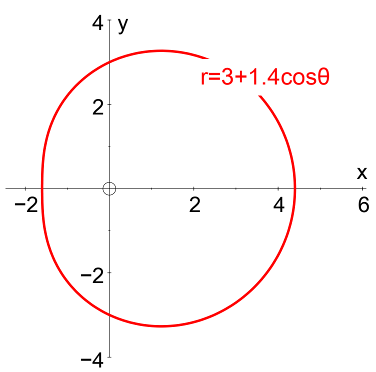
The sketches in these notes will show when r is negative by plotting a dotted line; these sections should be ignored as far as Edexcel A-level is concerned.

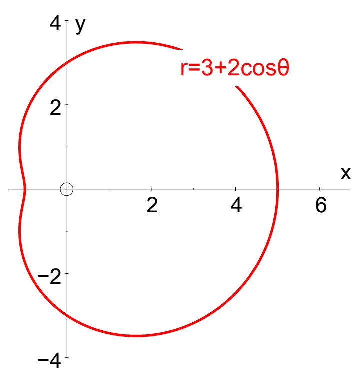
Some common curves

$\boldsymbol{r = a + b\cos\theta}$

Cardiod	**Limacon without dimple**	**Limacon with a dimple**
$a = b$	$a \ge 2b$,	$b \le a < 2b$



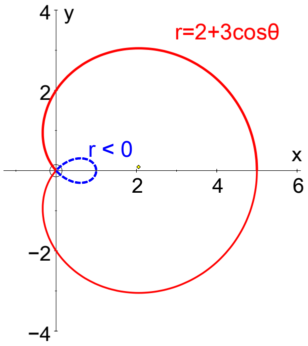




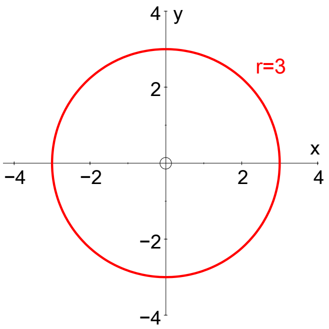
Limacon with a loop

$a < b$

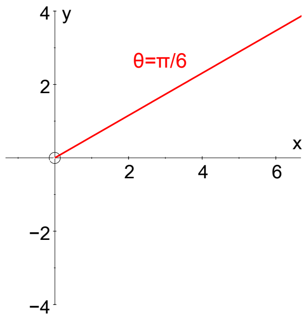
r negative in the loop



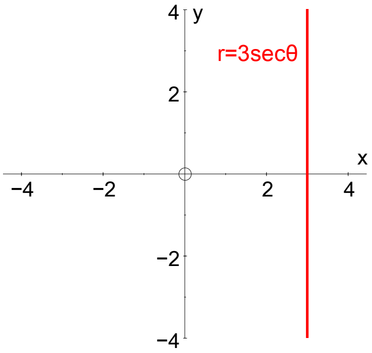
Circle



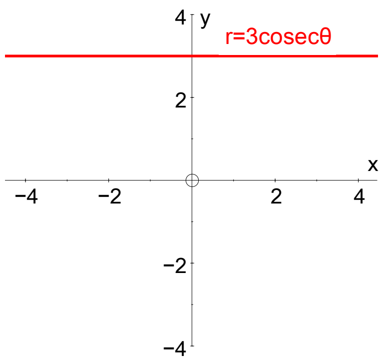
Half line



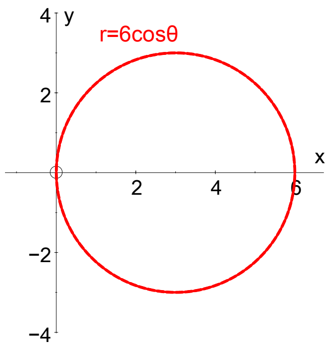
Line (x = 3)



Line (y = 3)



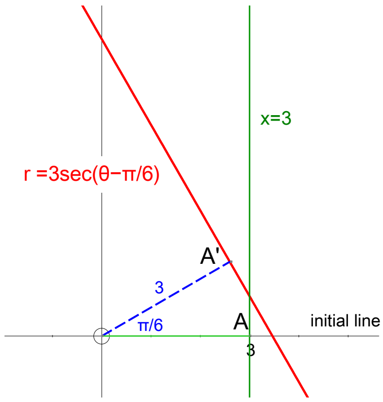
Circle



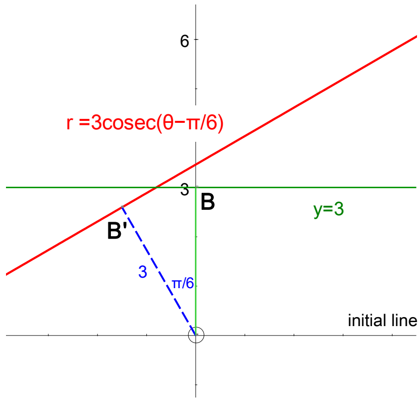
With Cartesian coordinates the graph of $y = f(x - a)$ is the graph of $y = f(x)$ translated through a in the x-direction.

In a similar way the graph of $\mathbf{r = 3\sec(\theta - \alpha)}$, or $\mathbf{r = 3\sec(\alpha - \theta)}$, is a rotation of the graph of $r = \sec\theta$ through α, anti-clockwise.

Line (x = 3 rotated through $\frac{\pi}{6}$)

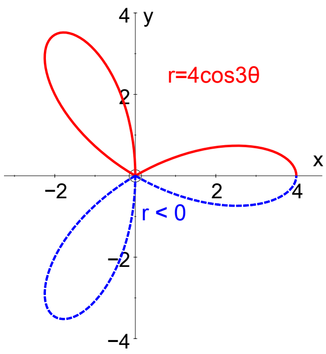


Line (y = 3 rotated through $\frac{\pi}{6}$)



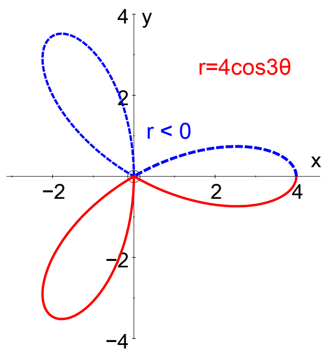
Rose Curves

$r = 4\cos 3\theta$
$0 \le \theta < \pi$



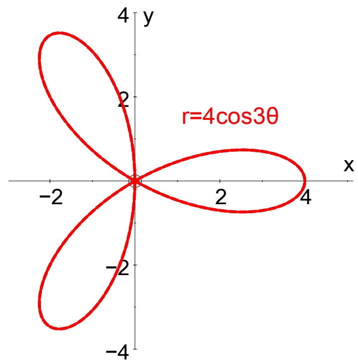
below x-axis, r negative

$r = 4\cos 3\theta$
$\pi \le \theta < 2\pi$



above x-axis, r negative

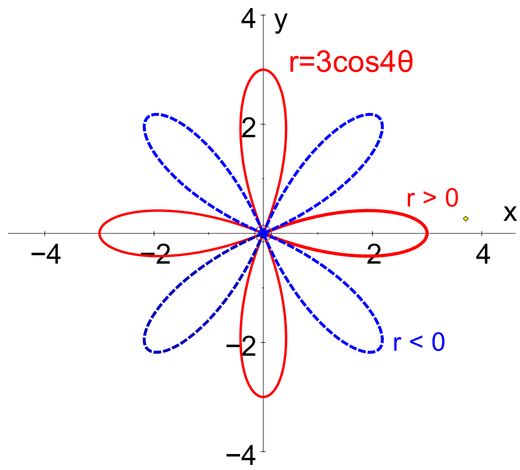
$r = 4\cos 3\theta$
$0 \le \theta < 2\pi$



whole curve for $r \ge 0$

The rose curve will always have n petals when n is odd, for $0 \le \theta < 2\pi$.

$r = 3\cos 4\theta$

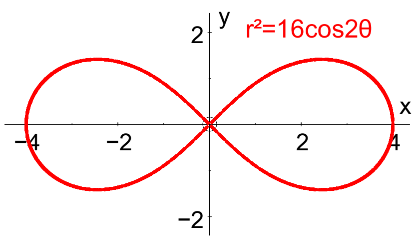


When n is *even* there will be n petals for $r \ge 0$ and $0 \le \theta < 2\pi$.

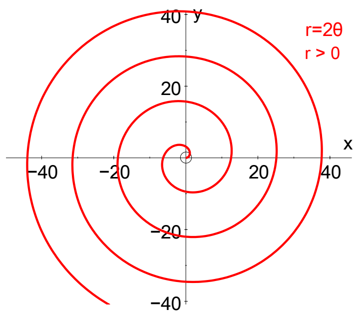
Thus, whether n is *odd* or *even*, the rose curve $r = a\cos\theta$ always has **n petals**, when **only the positive (or 0) values of r are taken.**

Edexcel only allow positive or 0 values of r.

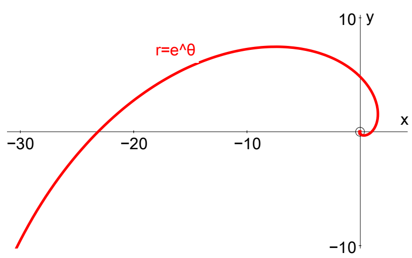
Leminiscate of Bernoulli



Spiral $r = 2\theta$

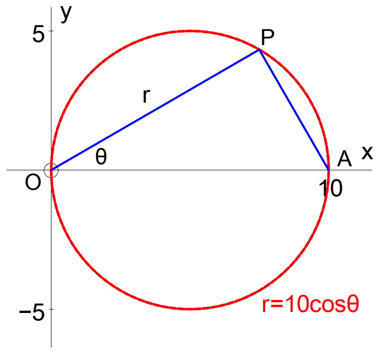


Spiral $r = e^{\theta}$



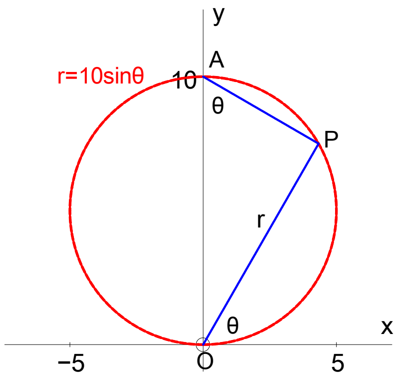
Circle $r = 10\cos\theta$

Notice that in the circle on OA as diameter, the angle P is 90° (angle in a semi-circle) and trigonometry gives us that $r = 10\cos\theta$.



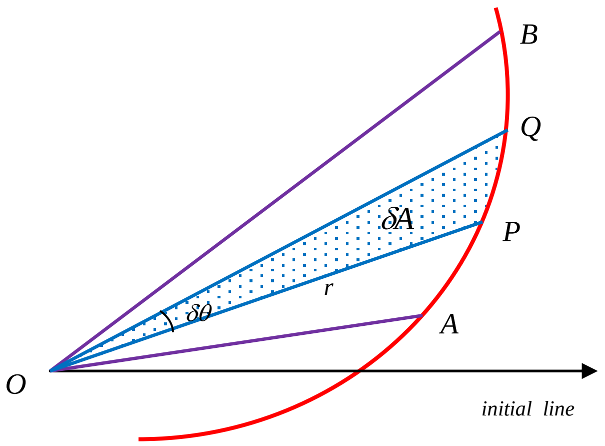
Circle $r = 10\sin\theta$

In the same way $r = 10\sin\theta$ gives a circle on the y-axis.



Areas using polar coordinates

Remember: area of a sector is $\frac{1}{2}r^2\theta$



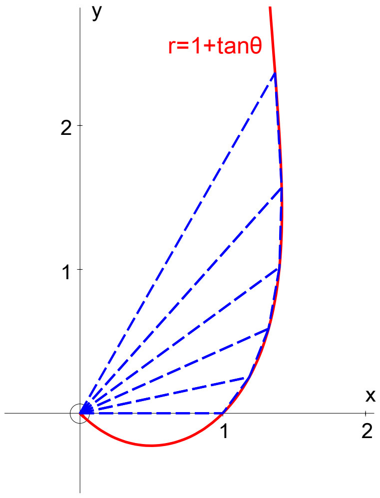
Area of $OPQ = \delta A \approx \frac{1}{2}r^2\delta\theta$

$\Rightarrow$ Area $OAB \approx \sum\left(\frac{1}{2}r^2\delta\theta\right)$

as $\delta\theta \to 0$

$\Rightarrow$ Area $OAB = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2\, d\theta$

Example: Find the area between the curve $r = 1 + \tan\theta$ and the half lines $\theta = 0$ and $\theta = \frac{\pi}{3}$



Solution: Area $= \int_0^{\pi/3} \frac{1}{2} r^2\, d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(1 + \tan\theta)^2\, d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(1 + 2\tan\theta + \tan^2\theta)\, d\theta$

$= \int_0^{\pi/3} \frac{1}{2}(2\tan\theta + \sec^2\theta)\, d\theta$

$= \frac{1}{2}\left[2\ln(\sec\theta) + \tan\theta\right]_0^{\pi/3}$

$= \ln 2 + \frac{\sqrt{3}}{2}$

Tangents parallel and perpendicular to the initial line

$y = r\sin\theta$ and $x = r\cos\theta$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

1) Tangents will be parallel to the initial line ($\theta = 0$), or horizontal, when $\frac{dy}{dx} = 0$

$\Rightarrow \quad \frac{dy}{d\theta} = 0$

$\Rightarrow \quad \frac{d}{d\theta}(r\sin\theta) = 0$

2) Tangents will be perpendicular to the initial line ($\theta = 0$), or vertical, when $\frac{dy}{dx}$ is infinite

$\Rightarrow \quad \frac{dx}{d\theta} = 0$

$\Rightarrow \quad \frac{d}{d\theta}(r\cos\theta) = 0$

Note that if both $\frac{dy}{d\theta} = 0$ and $\frac{dx}{d\theta} = 0$, then $\frac{dy}{dx}$ is not defined, and you should look at a sketch to help (or use l'Hôpital's rule).

Example: Find the coordinates of the points on $r = 1 + \cos\theta$ where the tangents are

(a) parallel to the initial line,

(b) perpendicular to the initial line.

Solution: $r = 1 + \cos\theta$ is shown in the diagram.

(a) Tangents parallel to $\theta = 0$ (horizontal)

$\Rightarrow \quad \frac{dy}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(r\sin\theta) = 0$

$\Rightarrow \quad \frac{d}{d\theta}((1+\cos\theta)\sin\theta) = 0 \qquad \Rightarrow \quad \frac{d}{d\theta}(\sin\theta + \sin\theta\cos\theta) = 0$

$\Rightarrow \quad \cos\theta - \sin^2\theta + \cos^2\theta = 0 \qquad \Rightarrow \quad 2\cos^2\theta + \cos\theta - 1 = 0$

$\Rightarrow \quad (2\cos\theta - 1)(\cos\theta + 1) = 0 \qquad \Rightarrow \quad \cos\theta = \frac{1}{2} \text{ or } -1$

$\Rightarrow \quad \theta = \pm\frac{\pi}{3} \quad \text{or} \quad \pi$

(b) Tangents perpendicular to $\theta = 0$ (vertical)

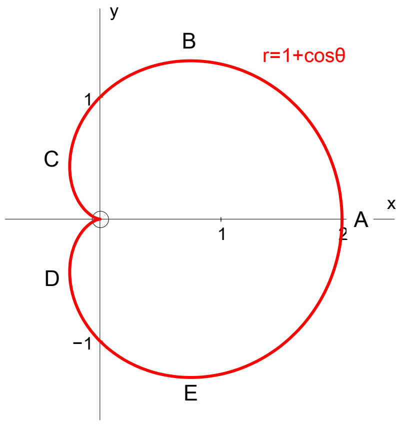
$\Rightarrow \quad \frac{dx}{d\theta} = 0 \quad \Rightarrow \quad \frac{d}{d\theta}(r\cos\theta) = 0$

$\Rightarrow \quad \frac{d}{d\theta}((1+\cos\theta)\cos\theta) = 0 \qquad \Rightarrow \quad \frac{d}{d\theta}(\cos\theta + \cos^2\theta) = 0$

$\Rightarrow \quad -\sin\theta - 2\cos\theta\sin\theta = 0 \qquad \Rightarrow \quad \sin\theta\,(1 + 2\cos\theta) = 0$

$\Rightarrow \quad \cos\theta = -\frac{1}{2} \text{ or } \sin\theta = 0$

$\Rightarrow \quad \theta = \pm\frac{2\pi}{3} \quad \text{or} \quad 0, \pi$



From the above we can see that

(a) the tangent is parallel to $\theta = 0$

at $B\left(\theta = \frac{\pi}{3}\right)$, and $E\left(\theta = -\frac{\pi}{3}\right)$,

also at $\theta = \pi$, the origin – see below (c)

(b) the tangent is perpendicular to $\theta = 0$

at $A\ (\theta = 0)$, $C\left(\theta = \frac{2\pi}{3}\right)$ and $D\left(\theta = \frac{-2\pi}{3}\right)$

(c) we also have both $\frac{dx}{d\theta} = 0$ and $\frac{dy}{d\theta} = 0$ when $\theta = \pi$!!!

From the graph it looks as if the tangent is parallel to $\theta = 0$ at the origin, when $\theta = \pi$, and from l'Hôpital's rule it can be shown that this is true.

Appendix

n^{th} roots of 1

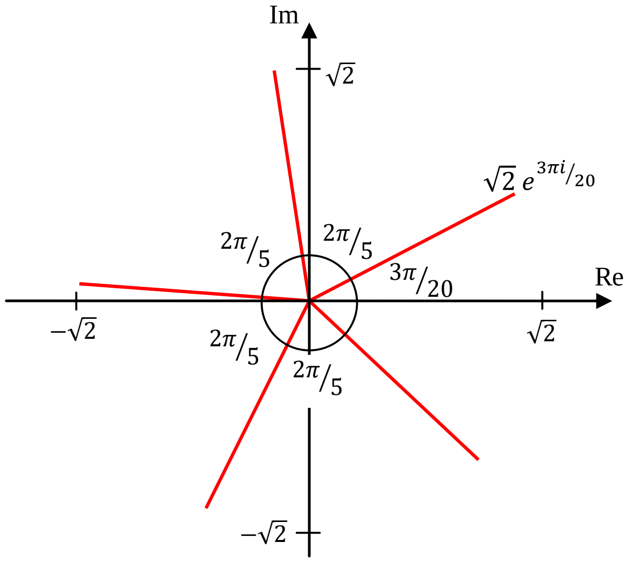
Short method

Example: Find the 5^{th} roots of $-4 + 4i = 4\sqrt{2}\, e^{3\pi i/4}$

Solution: First find the root with the smallest argument

$$\left(4\sqrt{2}\, e^{3\pi i/4}\right)^{1/5} = \sqrt{2}\, e^{3\pi i/20}$$

Then sketch the symmetrical 'spider' diagram where the angle between successive roots is $2\pi/5 = 8\pi/20$



then find all five roots by successively adding $8\pi/20$ to the argument of each root

to give

$\sqrt{2}\, e^{3\pi i/20}$, $\sqrt{2}\, e^{11\pi i/20}$, $\sqrt{2}\, e^{19\pi i/20}$,

$\sqrt{2}\, e^{27\pi i/20} = \sqrt{2}\, e^{-13\pi i/20}$, and $\sqrt{2}\, e^{35\pi i/20} = \sqrt{2}\, e^{-5\pi i/20}$.

This can be generalized to find the n^{th} roots of any complex number, adding $2\pi/n$ successively to the argument of each root.

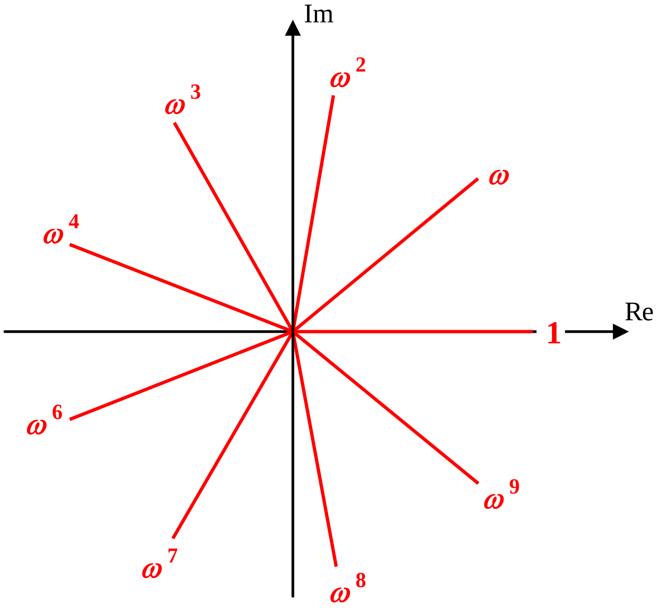
Warning: You must make sure that your method is very clear in an examination.

Sum of n^{th} roots of 1

Consider the solutions of $z^{10} = 1$, the complex 10^{th} roots of 1.

Suppose that ω is the complex 10^{th} root of 1 with the smallest argument. The 'spider' diagram shows that the roots are $\omega, \omega^2, \omega^3, \omega^4, \ldots, \omega^9$ and 1.

Symmetry indicates that the sum of all these roots is a real number, but to prove that this sum is 0 requires algebra.



$\omega \neq 1$, and $\omega^{10} = 1$

$\Rightarrow \quad 1 - \omega^{10} = 0$

$\Rightarrow \quad (1-\omega)(1 + \omega + \omega^2 + \omega^3 + \omega^4 + \ldots + \omega^9) = 0$ — factorising

$\Rightarrow \quad 1 + \omega + \omega^2 + \omega^3 + \omega^4 + \ldots + \omega^9 = 0,$ — since $1 - \omega \neq 0$

$\Leftrightarrow$ the sum of the complex 10^{th} roots of 1 is 0.

This can be generalized to show that the sum of the n^{th} roots of 1 is 0, for any n.

1^{st} order differential equations

Justification of the Integrating Factor method.

$\frac{dy}{dx} + Py = Q$ where P and Q are functions of x only.

We are looking for an Integrating Factor, R (a function of x), so that multiplication by R of the L.H.S. of the differential equation gives an exact derivative.

Multiplying the L.H.S. by R gives

$$R\frac{dy}{dx} + RPy$$

If this is to be an **exact** derivative we can see, by looking at the first term, that we should try

$$\frac{d}{dx}(Ry) = R\frac{dy}{dx} + y\frac{dR}{dx} = R\frac{dy}{dx} + RPy$$

$$\Rightarrow \quad y\frac{dR}{dx} = RPy$$

$$\Rightarrow \quad \int \frac{1}{R}\,dR = \int P\,dx$$

$$\Rightarrow \quad \ln R = \int P\,dx$$

$$\Rightarrow \quad R = e^{\int P\,dx}$$

Thus $e^{\int P\,dx}$ is the required I.F., Integrating Factor.

Linear 2nd order differential equations

Justification of the A.E. – C.F. technique for unequal roots

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

without loss of generality we can take the coefficient of $\frac{d^2y}{dx^2}$ *as* 1.

Let the roots of the A.E. be α and β $(\alpha \neq \beta)$, then the A.E. can be written as

$(m - \alpha)(m - \beta) = 0 \quad \Leftrightarrow \quad m^2 - (\alpha + \beta)\, m + \alpha\beta = 0$

So the differential equation can be written

$$\frac{d^2y}{dx^2} - (\alpha + \beta)\frac{dy}{dx} + \alpha\beta y = 0 \qquad \textbf{I}$$

We can 'sort of factorise' this to give

$$\left(\frac{d}{dx} - \alpha\right)\left(\frac{dy}{dx} - \beta y\right) = 0 \qquad \textbf{II} \qquad \text{'multiply' out to check}$$

Now put $\left(\frac{dy}{dx} - \beta y\right) = z$, in **II**, and we get $\frac{dz}{dx} - \alpha z = 0$

$$\Rightarrow \quad \int \frac{1}{z}\, dz = \int \alpha\, dx \quad \Rightarrow \quad z = A\, e^{\alpha x}$$

But $\left(\frac{dy}{dx} - \beta y\right) = z \quad \Rightarrow \quad \frac{dy}{dx} - \beta y = A\, e^{\alpha x}$

The Integrating Factor is $e^{-\beta x}$

$$\Rightarrow \quad e^{-\beta x}\frac{dy}{dx} - \beta e^{-\beta x} y = A\, e^{\alpha x} e^{-\beta x} \quad \Rightarrow \quad \frac{d\left(e^{-\beta x} y\right)}{dx} = A\, e^{(\alpha - \beta)x}$$

$$\Rightarrow \quad e^{-\beta x} y = \frac{A}{(\alpha - \beta)}\, e^{(\alpha - \beta)x} + B$$

$$\Rightarrow \quad y = A'\, e^{\alpha x} + B e^{\beta x}$$

which is the C.F., for **unequal** roots of the A.E.

Justification of the A.E. – C.F. technique for equal roots

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

without loss of generality we can take the coefficient of $\frac{d^2y}{dx^2}$ as 1.

Let the roots of the A.E. be α and α, (**equal** roots) then the A.E. can be written as

$(m - \alpha)(m - \alpha) = 0 \quad \Leftrightarrow \quad m^2 - 2\alpha m + \alpha^2 = 0$

So the differential equation can be written

$$\frac{d^2y}{dx^2} - 2\alpha\frac{dy}{dx} + \alpha^2 y = 0 \qquad \textbf{I}$$

We can 'sort of factorise' this to give

$$\left(\frac{d}{dx} - \alpha\right)\left(\frac{dy}{dx} - \alpha y\right) = 0 \qquad \textbf{II} \qquad \text{'multiply' out to check}$$

Now put $\left(\frac{dy}{dx} - \alpha y\right) = z$, in **II**, and we get $\frac{dz}{dx} - \alpha z = 0$

$$\Rightarrow \quad \int \frac{1}{z}\,dz = \int \alpha\,dx \quad \Rightarrow \quad z = A\,e^{\alpha x}$$

But $\left(\frac{dy}{dx} - \alpha y\right) = z \quad \Rightarrow \quad \frac{dy}{dx} - \alpha y = A\,e^{\alpha x}$

The Integrating Factor is $e^{-\alpha x}$

$$\Rightarrow \quad e^{-\alpha x}\frac{dy}{dx} - \alpha e^{-\alpha x} y = A\,e^{\alpha x} e^{-\alpha x} \quad \Rightarrow \quad \frac{d(e^{-\alpha x} y)}{dx} = A$$

$$\Rightarrow \quad e^{-\alpha x} y = Ax + B$$

$$\Rightarrow \quad y = (Ax + B)e^{\alpha x}$$

which is the C.F., for **equal** roots of the A.E.

Justification of the A.E. – C.F. technique for complex roots

Suppose that α and β are complex roots of the A.E., then they must occur as a conjugate pair (see FP1),

$\Rightarrow$ $\alpha = a + ib$ and $\beta = a - ib$

$\Rightarrow$ C.F. is $y = A\,e^{(a+ib)x} + B\,e^{(a-ib)x}$ assuming that calculus works for complex nos. which it does

$\Rightarrow$ $y = e^{ax}(A\,e^{ibx} + B\,e^{-ibx}) = e^{ax}(A(\cos x + i\sin x) + B(\cos x - i\sin x))$

$\Rightarrow$ C.F. is $y = e^{ax}(C\cos x + D\sin x)$, where C and D are arbitrary constants.

We now have the rules for finding the C.F. as before

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$ where a, b and c are constants.

First write down the Auxiliary Equation, A.E

A.E. $am^2 + bm + c = 0$

and solve to find the roots $m = \alpha$ or β

- If α and β are both real numbers, and if $\alpha \neq \beta$
  then the Complimentary Function, C.F., is
- $y = A\,e^{\alpha x} + Be^{\beta x}$, where A and B are arbitrary constants of integration

- If α and β are both real numbers, and if $\alpha = \beta$
  then the Complimentary Function, C.F., is
- $y = (A + Bx)\,e^{\alpha x}$, where A and B are arbitrary constants of integration

- If α and β are both complex numbers, and if $\alpha = a + ib,\ \beta = a - ib$
  then the Complimentary Function, C.F.,
- $y = e^{ax}(A\sin bx + B\cos bx)$,
  where A and B are arbitrary constants of integration

Justification that G.S. = C.F. + P.I.

Consider the differential equation $ay'' + by' + cy = f(x)$

Suppose that u (a function of x) is any member of the Complimentary Function,
and that v (a function of x) is a Particular Integral of the above D.E.

$\Rightarrow \quad au'' + bu' + cu = 0$

and $\quad av'' + bv' + cv = f(x)$

Let $w = u + v$

then $\quad aw'' + bw' + cw = a(u + v)'' + b(u + v)' + c(u + v)$

$$= (au'' + bu' + cu) + (av'' + bv' + cv) = 0 + f(x) = f(x)$$

$\Rightarrow \quad w$ is a solution of $ay'' + by' + cy = f(x)$

$\Rightarrow \quad$ **all** possible solutions $y = u + v$ are part of the General Solution. **I**

We now have to show that **any** member of the G.S. can be written in the form $u + v$, where u is some member of the C.F., and v is the P.I. used above.

Let z be **any** member of the G.S, then $az'' + bz' + cz = f(x)$.

Consider $z - v$

$$a(z - v)'' + b(z - v)' + c(z - v) = (az'' + bz' + cz) - (av'' + bv' + cv) = f(x) - f(x) = 0$$

$\Rightarrow \quad (z - v)$ is some member of the C.F. – call it u

$\Rightarrow \quad z - v = u \;\Rightarrow\; z = u + v$

thus **any** member, z, of the G.S. can be written in the form $u + v$, where u is some member of the C.F., and v is the P.I. used above. **II**

I and **II** $\Rightarrow$ the Complementary Function + a Particular Integral forms the complete General Solution.

Maclaurin's Series

Proof of Maclaurin's series

To express any function as a power series in x

Let $f(x) = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \dots$ **I**

put $x = 0$ $\Rightarrow$ $f(0) = a$

$\frac{d}{dx}$ $\Rightarrow$ $f'(x) = b + 2cx + 3dx^2 + 4ex^3 + 5fx^4 + \dots$

put $x = 0$ $\Rightarrow$ $f'(0) = b$

$\frac{d}{dx}$ $\Rightarrow$ $f''(x) = 2 \times 1c + 3 \times 2dx + 4 \times 3ex^2 + 5 \times 4fx^3 + \dots$

put $x = 0$ $\Rightarrow$ $f''(0) = 2 \times 1c$ $\Rightarrow$ $c = \frac{1}{2!}f''(0)$

$\frac{d}{dx}$ $\Rightarrow$ $f'''(x) = 3 \times 2 \times 1d + 4 \times 3 \times 2ex + 5 \times 4 \times 3fx^2 + \dots$

put $x = 0$ $\Rightarrow$ $f'''(0) = 3 \times 2 \times 1d$ $\Rightarrow$ $d = \frac{1}{3!}f'''(0)$

continuing in this way we see that the coefficient of x^n in **I** is $\frac{1}{n!}f^n(0)$

$$\Rightarrow \quad f(x) = f(0) + x f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^n(0) + \dots$$

The range of x for which this series converges depends on $f(x)$, and is beyond the scope of this course.

Proof of Taylor's series

If we put $f(x) = g(x + a)$ then

$$f(0) = g(a),\ f'(0) = g'(a),\ f''(0) = g''(a),\ \dots,\ f^n(0) = g^n(a), \dots$$

and Maclaurin's series becomes

$$g(x + a) = g(a) + x\, g'(a) + \frac{x^2}{2!} g''(a) + \frac{x^3}{3!} g'''(a) + \dots + \frac{x^n}{n!} g^n(a) + \dots$$

which is Taylor's series for $g(x + a)$ as a power series in x

Replace x by $(x - a)$ and we get

$$g(x) = g(a) + (x - a)\, g'(a) + \frac{(x-a)^2}{2!} g''(a) + \frac{(x-a)^3}{3!} g'''(a) + \dots + \frac{(x-a)^n}{n!} g^n(a) + \dots$$

which is Taylor's series for $g(x)$ as a power series in $(x - a)$

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