REVISION SHEET – FP2 (EDEXCEL)

INEQUALITIES

The main ideas are:

- Solving Inequalities

Before the exam you should know:

- When you solve an inequality, try substituting a few of the values for which you are claiming it is true back into the original inequality as a check.

Solving Inequalities

Broadly speaking inequalities can be solved in one of two ways, or sometimes in a combination of more than one of these ways.

Method 1

Use algebra to find an equivalent inequality which is easier to solve. When dealing with inequalities remember there are certain rules which need to be obeyed when performing algebraic manipulations. The main one is "DON'T MULTIPLY BY A NUMBER UNLESS YOU KNOW IT'S SIGN, IF IT'S NEGATIVE YOU MUST REVERSE THE INEQUALITY SIGN, IF IT'S POSITIVE THEN LEAVE THE INEQUALITY SIGN AS IT IS." For example, don't multiply by $(x-2)$ because that's positive when $x > 2$ and negative when $x < 2$. On the other hand $(x-2)^2$ is always positive so you can safely multiply by this (with no need to reverse the inequality sign).

Method 2

Sometimes is easier to rearrange an inequality of the from $\mathrm{g}(x) \le \mathrm{f}(x)$ to $\mathrm{g}(x) - \mathrm{f}(x) \le 0$ (you don't have to worry about reversing the inequality for such a rearrangement). Identify points where $\mathrm{g}(x) - \mathrm{f}(x) = 0$ or where $\mathrm{g}(x) - \mathrm{f}(x)$ has a vertical asymptote. Finally test whether the inequality is true in the various regions between these points.

Example Solve the inequality $3x - 2 \le \dfrac{x+2}{x-1}$

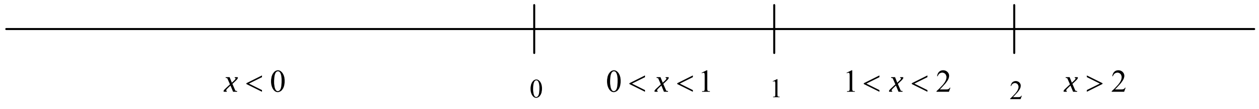
Solution (using Method 3)

$$3x - 2 \le \frac{x+2}{x-1} \Leftrightarrow 3x - 2 - \frac{x+2}{x-1} \le 0$$

$$\Leftrightarrow \frac{(3x-2)(x-1)-(x+2)}{x-1} \le 0 \Leftrightarrow \frac{3x^2-6x}{x-1} \le 0$$

$$\Leftrightarrow \frac{3x(x-2)}{x-1} \le 0$$

Looking at the expression, $3x = 0$ if $x = 0$, $x - 2 = 0$ if $x = 2$ and $x - 1 = 0$ if $x = 1$.
This means that the truth of the inequality should be tested in each of the following regions



It can be seen that the inequality is TRUE if $x < 0$, false if $0 < x < 1$, TRUE if $1 < x < 2$ and FALSE if $x > 2$. The solution is therefore $x \le 0$, $1 < x \le 2$. Can you see why $x = 0$ and $x = 2$ are included as values for which the inequality is true, but $x = 1$ is not?

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SERIES

The main ideas are:

- Summing Series using standard formulae
- Telescoping

Before the exam you should know:

- The standard formula:

$$\sum_{r=1}^{n} r, \sum_{r=1}^{n} r^2, \sum_{r=1}^{n} r^3$$

- And be able to spot that a series like

$$(1\times 2)+(2\times 3)+...+n(n+1)$$

can be written in sigma notation as:

$$\sum_{r=1}^{n} r(r+1)$$

Summing Series

Using standard formulae

Fluency is required in manipulating and simplify standard formulae sums like:

$$\sum_{r=1}^{n} r\left(r^2+1\right)=\sum_{r=1}^{n} r^3+\sum_{r=1}^{n} r=\frac{n^2\left(n+1\right)^2}{4}+\frac{n\left(n+1\right)}{2}$$

$$=\frac{1}{4}n\left(n+1\right)\left[n\left(n+1\right)+2\right]$$

$$=\frac{1}{4}n\left(n+1\right)\left(n^2+n+2\right).$$

The Method of Differences (Telescoping)

Since $\frac{r+4}{r(r+1)(r+2)}=\frac{2}{r}-\frac{3}{r+1}+\frac{1}{r+2}$ (frequently in exam questions you are told to show that this is true first) it is possible to demonstrate that:

$$\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}=\left(2-\frac{3}{2}+\frac{1}{3}\right)+\left(\frac{2}{2}-\frac{3}{3}+\frac{1}{4}\right)+\left(\frac{2}{3}-\frac{3}{4}+\frac{1}{5}\right)+...$$

$$+\left(\frac{2}{n-2}-\frac{3}{n-1}+\frac{1}{n}\right)+\left(\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}\right)+\left(\frac{2}{n}-\frac{3}{n+1}+\frac{1}{n+2}\right)$$

In this kind of expression many terms cancel with each other. For example, the (+)$\frac{1}{3}$ in the first bracket cancels with the (–)$\frac{3}{3}$ in the second bracket and the (+)$\frac{2}{3}$ in the third bracket. (subsequent fractions that are cancelling are doing so with terms in the "…" part of the sum.)

This leaves $\sum_{r=1}^{n}\frac{r+4}{r(r+1)(r+2)}=\frac{3}{2}-\frac{2}{n+1}+\frac{1}{n+2}$.

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DIFFERENTIAL EQUATIONS

The main ideas are:

- First Order Differential Equations
- Second Order Differential Equations

Before the exam you should know:

- How to solve first order differential equations by separating the variable and the using an integrating factor, including finding general and particular solutions.
- How to solve second order homogeneous and non-homogeneous differential equations, including finding general and particular solutions.

First Order Differential Equations

Separating the Variables

For a differential equation of the type $\frac{dy}{dx} = f(x)g(y)$ you should recognise that it can be solved by 'separating the variables', i.e. $\int \frac{1}{g(y)} dy = \int f(x) dx$.

Example Find the general solution to the differential equation $\frac{dy}{dx} = \frac{y+2}{x}$, where *x*, *y* > 0.

Solution $\frac{dy}{dx} = \frac{y+2}{x}$, $\int \frac{1}{y+2} dy = \int \frac{1}{x} dx$, $\ln(y+2) = \ln x + c$.

This is the general solution but you may be asked to rearrange to give *y* in terms of *x*, which will require you to manipulate logs and exponentials. Here it would give $y = Ax - 2$, where *A* is a constant.

Integrating Factor

For a differential equation of the type $\frac{dy}{dx} + Py = Q$ where *P* and *Q* are functions of *x* you should recognise that it can be solved by using an 'integrating factor', i.e. $e^{\int P dx}$. It is important to remember that it has to be in the correct form before you use the method. Similarly, you may be expected to use a given substitution to get a differential equation into the form suitable for using either of the two methods highlighted.

Example Find the particular solution to the differential equation $\frac{dy}{dx} + 2xy = x$, for which *y* = 4 when *x* = 0.

Solution This is already in the standard form so the Integrating factor = $e^{\int 2x dx} = e^{x^2}$
Multiplying through by the integrating factor gives

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2} \qquad \Rightarrow \frac{d}{dx}(e^{x^2}y) = xe^{x^2} \qquad \Rightarrow e^{x^2}y = e^{x^2} + c \qquad \Rightarrow y = 1 + ce^{-x^2}$$

Substituting *x* = 0 and *y* = 4 gives: $4 = 1 + c$ $\Rightarrow c = 3$

Thus the particular solution of the differential equation is: $y = 1 + 3e^{-x^2}$

Second Order Differential Equations

The auxiliary equation method

For equations of the form $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ the auxiliary method can be summarized as follows:

- Write down the auxiliary equation $am^2+bm+c=0$
- Solve the equation to obtain either two distinct roots, one repeated root, or two complex roots (these may be pure imaginary)
- Write down the general solution in the appropriate form, as seen in the following table:

Roots of auxiliary equation	Form of general solution
Two distinct roots α and β	$y=Ae^{\alpha x}+Be^{\beta x}$
One repeated root m	$y=(A+Bx)e^{mx}$
Pure imaginary roots $\pm n\,\mathbf{i}$	$y=A\cos nx+B\sin nx$
Complex roots $p\pm q\,\mathbf{i}$	$y=e^{px}(A\cos qx+B\sin qx)$

It would also be useful to detail the possible particular integrals that you may need when solving a differential equation that does not equal zero.

Function *f*(*x*)	Particular integral
Constant term	C
Linear function	$ax+b$
Quadratic function	ax^2+bx+c
Exponential function involving e^{px}	ke^{px}
Function involving $\cos px$ and / or $\sin px$	$a\cos px+b\sin px$

Note. In some special cases you will be required to use a non-standard particular integral, i.e. kxe^{px} instead of ke^{px}, please review your notes and textbooks if you require further detail.

Example Find the general solution of the differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x$.

Solution

The auxiliary equation is $m^2+2m+1=0$, $(m+1)^2=0$, $m=-1$

Therefore the complementary function is $y=(A+Bx)e^{-x}$

The particular integral is $y=ax+b$, $\frac{dy}{dx}=a$, $\frac{d^2y}{dx^2}=0$

Substituting into the differential equation: $0+2a+(ax+b)=x$

Equating coefficients of x: $a=1$
Equating constants: $2a+b=0$, so $b=-2$
The particular integral is $y=x-2$,

Hence the general solution is $y=(A+Bx)e^{-x}+x-2$
Note. You may also be asked to consider a particular solution for some conditions which are given.

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POLAR COORDINATES

The main ideas are:

- What Polar Coordinates are
- Conversion between Cartesian and Polar Coordinates
- Curves defined using Polar Coordinates
- Calculating areas for curves defined using Polar Coordinates

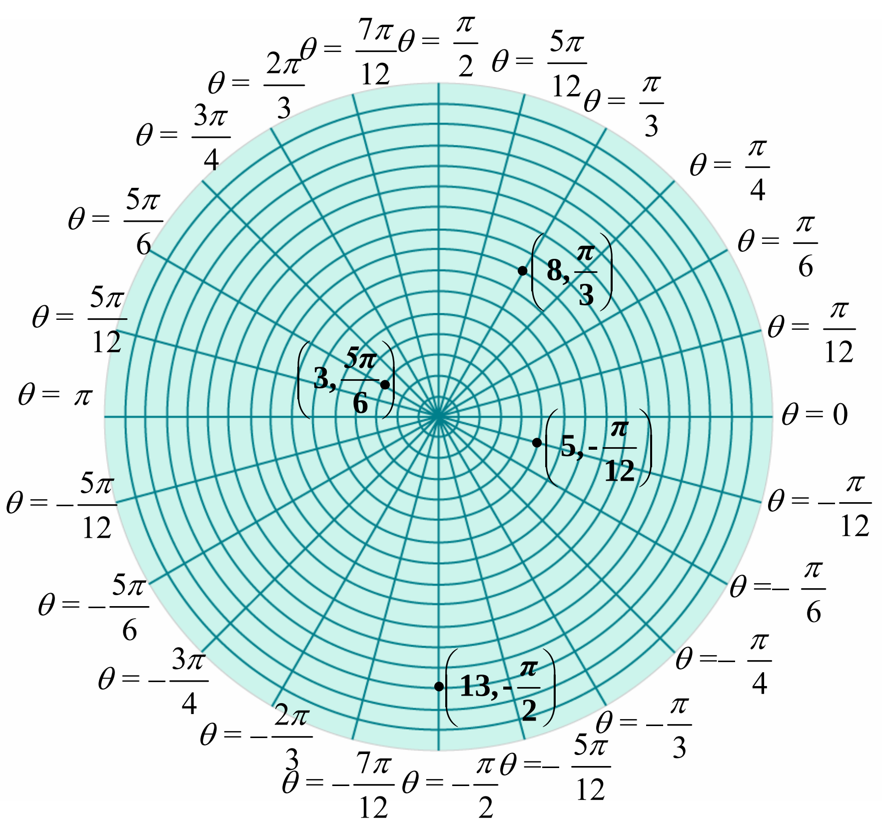
Before the exam you should know:

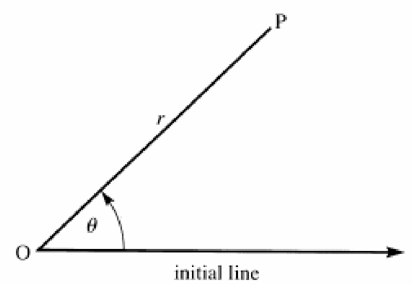
- How to change between polar coordinates (r, θ) and Cartesian coordinates (x, y) using $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$.
- You'll need to be very familiar with the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ and be able to give exact values of the trig functions for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$.
- How to sketch a curve given by a polar equation.
- The area of a sector is given by $\int_\alpha^\beta \frac{1}{2}r^2\,d\theta$.

How Polar Coordinates Work

You will be familiar with using Cartesian Coordinates (x, y) to specify the position of a point in the plane. Polar coordinates use the idea of describing the position of a point P by giving its distance r from the origin and the angle θ between OP and the positive x-axis. The angle θ is positive in the anticlockwise sense from the initial line. If it is necessary to specify the polar coordinates of a point uniquely then you use those for which $r > 0$ and $-\pi < \theta \le \pi$.

It is sometimes convenient to let r take negative values with the natural interpretation that (– r, θ) is the same as (r, $\theta + \pi$).





It is easy to change between polar coordinates (r, θ) and Cartesian coordinates (x, y) since $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$. You need to be careful to choose the right quadrant when finding θ, since the equation $\tan\theta = \frac{y}{x}$ always gives two values of θ, differing by π. Always draw a sketch to check which one of these is correct.

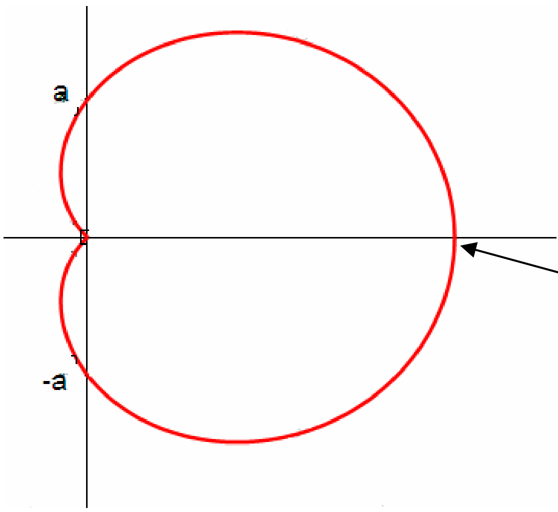
The Polar Equation of a Curve

The points (r, θ) for which the values of r and θ are linked by a function f form a curve whose polar equation is $r = f(\theta)$. A good way to draw a sketch of a curve is to calculate r for a variety of values of θ.

Example Sketch the curve which has polar equation $r = a(1+\sqrt{2}\cos\theta)$ for $-\frac{3}{4}\pi \le \theta \le \frac{3}{4}\pi$, where a is a positive constant.

Solution Begin by calculating the value or r for various values of θ. This is shown in the table. The curve can now be sketched.

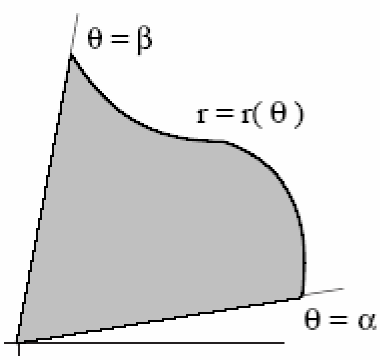
θ	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
r	0	a	$2a$	$a\left(1+\sqrt{2}\right)$	$2a$	a	0



It's a good exercise to try to spot the points given in the table above in polar coordinates on the curve shown here.

For example the point $\left(a\left(1+\sqrt{2}\right),0\right)$ is here.

The Area of a Sector



The area of the sector shown in the diagram is $\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$

Example

A curve has polar equation $r = a(1+\sqrt{2}\cos\theta)$ for $-\frac{3}{4}\pi \le \theta \le \frac{3}{4}\pi$, where a is a positive constant. Find the area of the region enclosed by the curve.

Solution The area is clearly twice the area of the sector given by

$$2\int_0^{\frac{3\pi}{4}} \frac{1}{2} r^2 d\theta = a^2 \int_0^{\frac{3\pi}{4}} \left(1+\sqrt{2}\cos\theta\right)^2 d\theta = a^2 \int_0^{\frac{3\pi}{4}} \left(1+2\sqrt{2}\cos\theta + 2\cos^2\theta\right) d\theta$$

$$= a^2 \int_0^{\frac{3\pi}{4}} \left(2+2\sqrt{2}\cos\theta + \cos 2\theta\right) d\theta$$

$$= a^2 \left[2\theta + 2\sqrt{2}\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{3\pi}{4}}$$

$$= \frac{3}{2}\left(\pi+1\right)a^2$$

Note Even though r can be negative for certain values of θ, $\frac{1}{2}r^2$ is always positive, so there is no problem of 'negative areas' as there is with curves below the x-axis in cartesian coordinates.

Be careful however when considering loops contained inside loops.