REVISION SHEET – FP2 (OCR)

CALCULUS

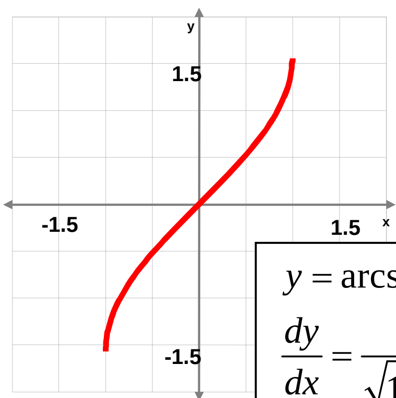
The main ideas are:

- Calculus using inverse trig functions & hyperbolic trig functions and their inverses.
- Maclaurin series

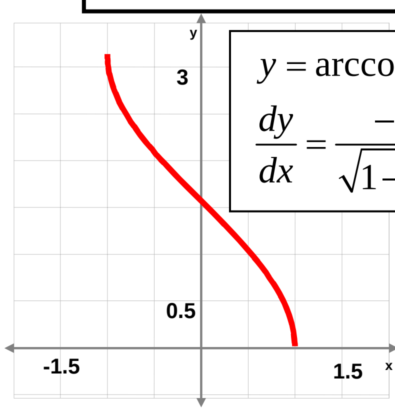
Before the exam you should know:

- That you can differentiate the trig functions, the hyperbolic trig functions and their inverses.
- That you can apply the standard rules for differentiation (product rule, quotient rule and chain rule) to functions which involve the above.
- That you can integrate trig functions and hyperbolic trig functions.
- That you can integrate, arcsin(x), arccos(x), arctan(x), arccot(x), arsinh(x), arcosh(x) etc using integration by parts.
- Your trig identities and hyperbolic function identities and how to use them in integration problems. Particularly get familiar with useful substitutions to make for certain problems.

Differentiating the Inverse Trig Functions

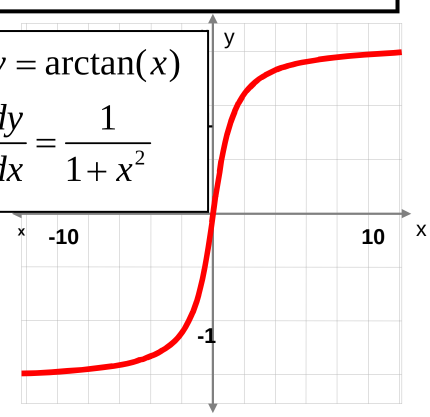


$$y = \arcsin(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$



$$y = \arccos(x) \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$y = \arctan(x) \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$



It is important to be aware of what the range is for each of these, namely:

$$-\frac{\pi}{2} \le \arcsin \le \frac{\pi}{2}, \qquad 0 \le \arccos \le \pi, \qquad -\frac{\pi}{2} \le \arctan \le \frac{\pi}{2}$$

Standard Calculus of Inverse Trig and Hyperbolic Trig Functions

$$y = \arcsin(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$y = \arccos(x) \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$y = \arctan(x) \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y = \operatorname{ar}\sinh(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$y = \operatorname{ar}\cosh(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}} = \operatorname{ar}\cosh\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2+a^2}} = \operatorname{ar}\sinh\left(\frac{x}{a}\right) + c$$

Calculus using these functions

The examples below are very typical and show most of the common tricks. Note – details of all substitutions have been omitted, make sure you understand how to do them in this case and also in the case of a definite integral.

- $\displaystyle\int \frac{1}{\sqrt{4x^2+16x+32}}dx = \frac{1}{2}\int \frac{1}{\sqrt{(x+2)^2+4}}dx = \frac{1}{2}\operatorname{arsinh}\left(\frac{x+2}{2}\right)+c$
- $\displaystyle\int \frac{4}{\sqrt{5+3x-9x^2}}dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{5}{9}-\left(x^2-\frac{x}{3}\right)}}dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{21}{36}-\left(x-\frac{1}{6}\right)^2}}dx = \frac{4}{3}\arcsin\left(\frac{6\left(x-\frac{1}{6}\right)}{\sqrt{21}}\right)+c = \frac{4}{3}\arcsin\left(\frac{6x-1}{\sqrt{21}}\right)+c$
- $\displaystyle\int \frac{3}{\sqrt{2x^2+4x-10}}dx = \frac{3}{\sqrt{2}}\int \frac{1}{\sqrt{(x+1)^2-6}}dx = \frac{3}{\sqrt{2}}\operatorname{arcosh}\left(\frac{x+1}{\sqrt{6}}\right)+c$
- $\displaystyle y = \operatorname{arcosh}(x^2) \Rightarrow \frac{dy}{dx} = \frac{2x}{\sqrt{x^4-1}}$ (to see this use the chain rule, set $z = x^2$ and then $\displaystyle\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$).

Some useful integration tricks

Splitting up an integration: e.g. $\displaystyle\int_1^5 \frac{x+5}{x^2+4}dx = \int_1^5 \frac{x}{x^2+4}dx + \int_1^5 \frac{5}{x^2+4}dx$

By inspection: e.g. Since $\ln(x^2+4)$ gives $\displaystyle\frac{2x}{x^2+4}$ when differentiated, we have $\displaystyle\int \frac{x}{x^2+4}dx = \frac{1}{2}\ln(x^2+4)+c$ or since $(x^2+1)^{\frac{1}{2}}$ gives $x(x^2+1)^{-\frac{1}{2}}$ when differentiated, we have $\displaystyle\int \frac{x}{\sqrt{x^2+1}}dx = \sqrt{x^2+1}+c$

Using clever substitutions: e.g. the substitution $u = \sinh(x)$ will help you with $\int \sqrt{x^2+1}dx$.

Maclaurin Series

The Maclaurin expansion for a function f(x) as far as the term in x^n looks as follows.

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^n}{n!}f^{(n)}(0)$$

The Maclaurin ***series*** is obtained by including infinitely many terms (i.e. not terminating the sum as above). It may only be valid for certain values of x. Examples include:

$\displaystyle\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$ which is valid for all x, $\displaystyle\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + ...$ which is valid only when $|x| < \frac{1}{2}$, note that this second example is the same as the binomial expansion of $(1-2x)^{-1}$.

Note. You can find the Maclaurin series of, e.g. f($2x$), by taking the series for f(x) and replacing the x's with $2x$.

Reduction Formulae

You should be able to derive and use reduction formulae for the evaluation of definite integrals in simple cases. i.e. to calculate $\int_0^1 x^4 e^x dx$.

REVISION SHEET – FP2 (OCR)

HYPERBOLIC TRIG FUNCTIONS

The main ideas are:

- Definitions of the hyperbolic trig functions and their inverses.
- Working with the hyperbolic trig functions.
- Identities involving hyperbolic trig functions.

Before the exam you should know:

- The definitions $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$, $\tanh(x)=\dfrac{\sinh(x)}{\cosh(x)}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$
- That you can prove that $\operatorname{arccosh}(x)=\ln(x+\sqrt{x^2-1})$, $\operatorname{arsinh}(x)=\ln(x+\sqrt{x^2+1})$ $\operatorname{artanh}(x)=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$
- Your trig identities and hyperbolic function identities, experience will tell you when it is best to work in the exponential form when dealing with equations.
- And be able to prove hyperbolic identities from the definitions $\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$, it's worth practicing indices for this.

The Hyperbolic Trig Functions

These are defined as:

$\sinh(x)=\dfrac{e^x-e^{-x}}{2}$, $\cosh(x)=\dfrac{e^x+e^{-x}}{2}$,

$\tanh(x)=\dfrac{\sinh(x)}{\cosh(x)}=\dfrac{e^x-e^{-x}}{e^x+e^{-x}}$.

For example, $\sinh(\ln 10)=\dfrac{e^{\ln 10}-e^{-\ln 10}}{2}=\dfrac{10-\frac{1}{10}}{2}=\dfrac{99}{20}$.

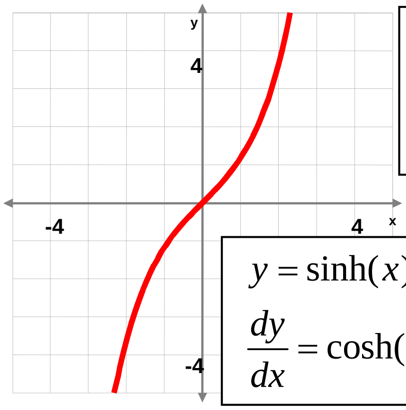
The Inverse Hyperbolic Trig Functions

Just as the hyperbolic trig functions are defined in terms of e^x, their inverses can be expressed in term of logs. In fact $\operatorname{arcosh}(x)=\ln(x+\sqrt{x^2-1})$, $\operatorname{arsinh}(x)=\ln(x+\sqrt{x^2+1})$, $\operatorname{artanh}(x)=\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$. You should be able to prove (and use) all of these. Here is the proof that $\operatorname{arcosh}(x)=\ln(x+\sqrt{x^2-1})$.

Let $y=\operatorname{ar}\cosh(x)$, then $x=\cosh(y)=\dfrac{e^y+e^{-y}}{2}$. Rearranging this gives $0=e^y-2x+e^{-y}$. Multiplying this by e^y gives $0=e^{2y}-2xe^y+1$. This is a quadratic in e^y and using the formula for the roots of a quadratic gives $e^y=\dfrac{2x\pm\sqrt{4x^2-4}}{2}=x\pm\sqrt{x^2-1}$. Taking logs gives $y=\operatorname{ar}\cosh(x)=\ln(x\pm\sqrt{x^2-1})$. Do you know why the expression with the minus sign is rejected here?

These expressions can be used to give exact values of the inverse hyperbolic trig functions in term of logs. For example, $\operatorname{arcosh}\left(\dfrac{5}{3}\right)=\ln\left(\dfrac{5}{3}+\sqrt{\left(\dfrac{5}{3}\right)^2-1}\right)=\ln\left(\dfrac{5}{3}+\sqrt{\dfrac{16}{9}}\right)=\ln(3)$.

Graphs of the Hyperbolic Trig Functions

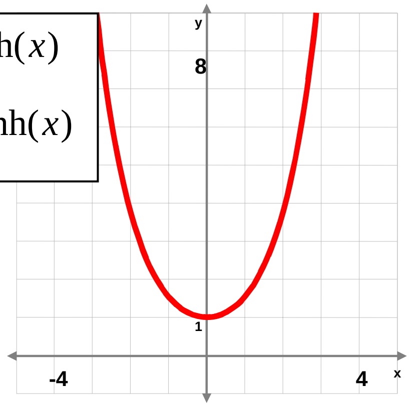


$y = \sinh(x)$

$\dfrac{dy}{dx} = \cosh(x)$

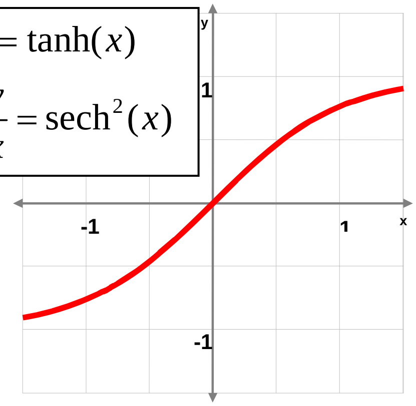
$y = \cosh(x)$

$\dfrac{dy}{dx} = \sinh(x)$

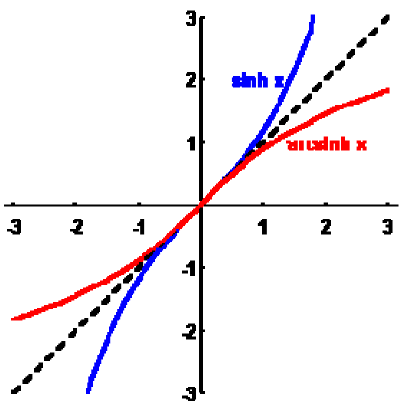


$y = \tanh(x)$

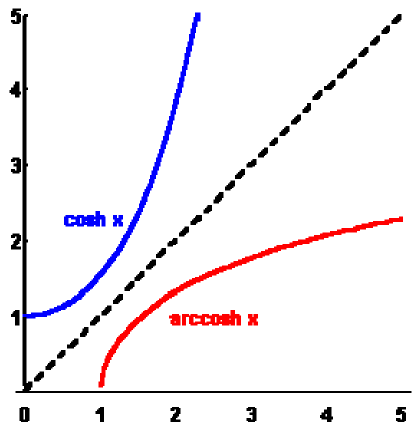
$\dfrac{dy}{dx} = \text{sech}^2(x)$



Graphs of the Inverse Hyperbolic Trig Functions



You must also know the graphs of the inverse hyperbolic trig functions, arsinh, arcosh and artanh. As for any function these are obtained by reflecting the respective graphs of sinh, cosh and tanh in $y = x$. The examples of arsinh and arcosh are shown here. Notice that arcosh(x) is only defined for x greater than or equal to 1.



Identities Involving Hyperbolic Trig Functions

Identities involving hyperbolic trig functions include:

$$\cosh^2 u - \sinh^2 u = 1, \quad \cosh(2u) = \cosh^2 u + \sinh^2 u, \quad \sin(u+v) = \sinh(u)\cosh(v) + \cosh(u)\sinh(v)$$

The only difference between a hyperbolic trig identity and the corresponding standard trig identity is that the sign is reversed when a product of two sines is replaced by a product of two sinhs. This is called Osborn's Rule.

You can prove any hyperbolic trig identity using their definitions and should be able to do this for the exam.

Equations Involving Hyperbolic Trig Functions

Example Solve the equation $13\cosh x + 5\sinh x = 20$ giving your answer in terms of natural logarithms.

Solution $13\cosh x + 5\sinh x = 20 \Rightarrow 13\left(\dfrac{e^x + e^{-x}}{2}\right) + 5\left(\dfrac{e^x - e^{-x}}{2}\right) = 20$

$$\Rightarrow 18e^x + 8e^{-x} - 40 = 0$$

$$\Rightarrow 9e^{2x} - 20e^x + 4 = 0 \Rightarrow \left(9e^x - 2\right)\left(e^x - 2\right) = 0$$

$$\Rightarrow e^x = \frac{2}{9} \text{ or } e^x = 2$$

$$\Rightarrow x = \ln\left(\frac{2}{9}\right) \text{or } x = \ln 2$$

Disclaimer: Every effort has gone into ensuring the accuracy of this document. However, the FM Network can accept no responsibility for its content matching each specification exactly.

REVISION SHEET – FP2 (OCR)

POLAR COORDINATES

The main ideas are:

- What Polar Coordinates are
- Conversion between Cartesian and Polar Coordinates
- Curves defined using Polar Coordinates
- Calculating areas for curves defined using Polar Coordinates

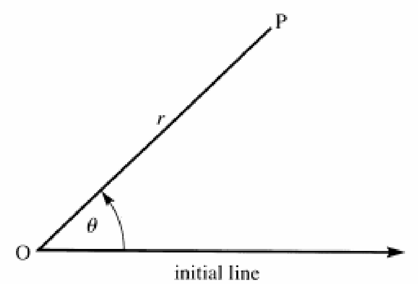
Before the exam you should know:

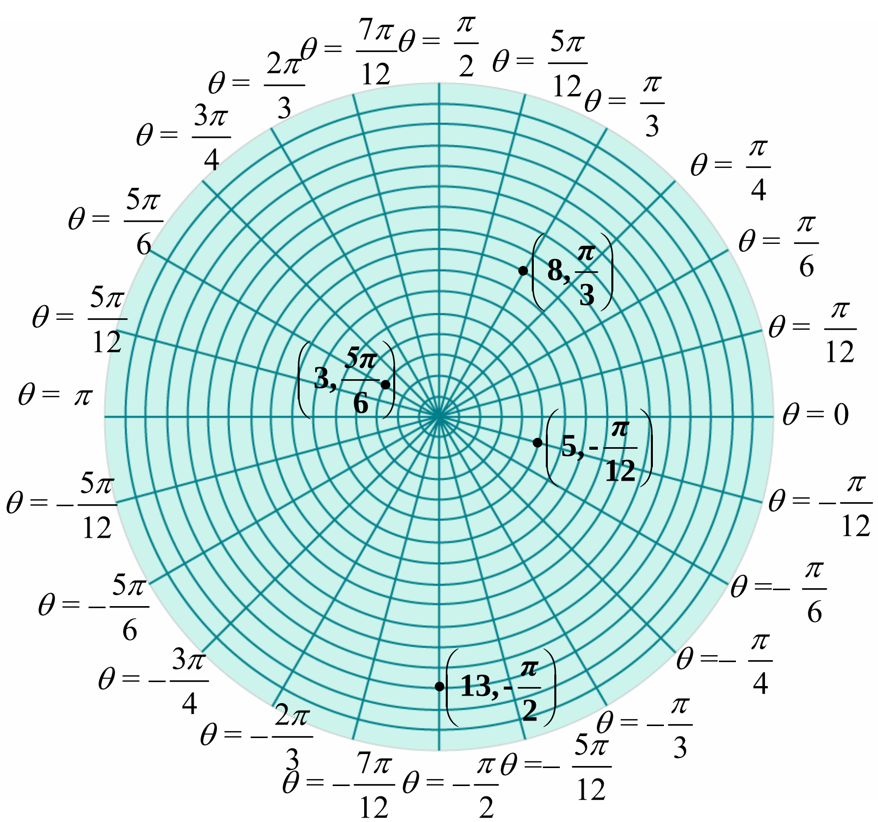
- How to change between polar coordinates (r, θ) and Cartesian coordinates (x, y) use $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$.
- You'll need to be very familiar with the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ and be able to give exact values of the trig functions for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$.
- How to sketch a curve given by a polar equation.
- The area of a sector is given by $\int_\alpha^\beta \frac{1}{2}r^2\,d\theta$.

How Polar Coordinates Work

You will be familiar with using Cartesian Coordinates (x, y) to specify the position of a point in the plane. Polar coordinates use the idea of describing the position of a point P by giving its distance r from the origin and the angle θ between OP and the positive x-axis. The angle θ is positive in the anticlockwise sense from the initial line. If it is necessary to specify the polar coordinates of a point uniquely then you use those for which $r > 0$ and $-\pi < \theta \le \pi$.

It is sometimes convenient to let r take negative values with the natural interpretation that ($-$r, θ) is the same as (r, $\theta + \pi$).





It is easy to change between polar coordinates (r, θ) and Cartesian coordinates (x, y) since $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$. You need to be careful to choose the right quadrant when finding θ, since the equation $\tan\theta = \frac{y}{x}$ always gives two values of θ, differing by π. Always draw a sketch to check which one of these is correct.

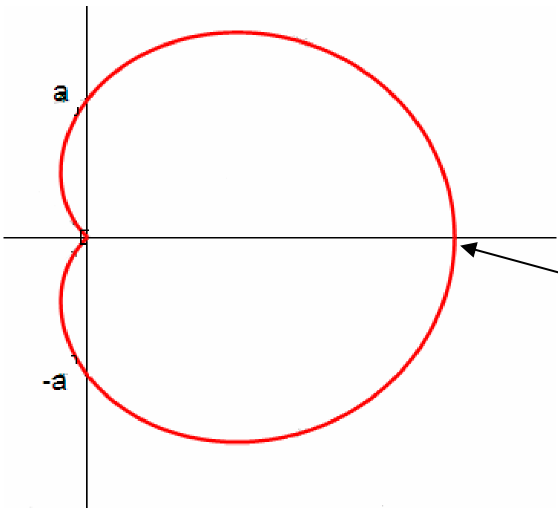
The Polar Equation of a Curve

The points (r, θ) for which the values of r and θ are linked by a function f form a curve whose polar equation is $r = \mathrm{f}(\theta)$. A good way to draw a sketch of a curve is to calculate r for a variety of values of θ.

Example Sketch the curve which has polar equation $r = a(1+\sqrt{2}\cos\theta)$ for $-\frac{3}{4}\pi \le \theta \le \frac{3}{4}\pi$, where a is a positive constant.

Solution Begin by calculating the value or r for various values of θ. This is shown in the table. The curve can now be sketched.

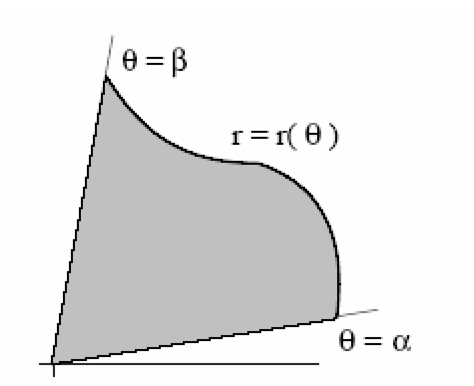
θ	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
r	0	a	$2a$	$a\left(1+\sqrt{2}\right)$	$2a$	a	0



It's a good exercise to try to spot the points given in the table above in polar coordinates on the curve shown here.

For example the point $\left(a\left(1+\sqrt{2}\right),0\right)$ is here.

The Area of a Sector



The area of the sector shown in the diagram is $\int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$

Example

A curve has polar equation $r = a(1+\sqrt{2}\cos\theta)$ for $-\frac{3}{4}\pi \le \theta \le \frac{3}{4}\pi$, where a is a positive constant. Find the area of the region enclosed by the curve.

Solution The area is clearly twice the area of the sector given by $0 \le \theta \le \frac{3}{4}\pi$. Therefore the area is

$$2\int_0^{\frac{3\pi}{4}} \frac{1}{2} r^2 d\theta = a^2 \int_0^{\frac{3\pi}{4}} \left(1+\sqrt{2}\cos\theta\right)^2 d\theta = a^2 \int_0^{\frac{3\pi}{4}} \left(1+2\sqrt{2}\cos\theta + 2\cos^2\theta\right) d\theta$$

$$= a^2 \int_0^{\frac{3\pi}{4}} \left(2+2\sqrt{2}\cos\theta+\cos 2\theta\right) d\theta$$

$$= a^2 \left[2\theta + 2\sqrt{2}\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{3\pi}{4}}$$

$$= \frac{3}{2}(\pi+1)a^2$$

Note Even though r can be negative for certain values of θ, $\frac{1}{2}r^2$ is always positive, so there is no problem of 'negative areas' as there is with curves below the x-axis in cartesian coordinates.

Be careful however when considering loops contained inside loops.

REVISION SHEET – FP2 (OCR)

GRAPHS

The main ideas are:

- Sketching Graphs of Rational Functions

Before the exam you should know:

- There are three main cases of horizontal asymptotes.
  - One is a curve which is a linear polynomial divided by a linear polynomial, for example $y = \frac{4x+1}{3x-2}$. This has a horizontal asymptote at $y = \frac{\text{coefficient of } x \text{ on the top}}{\text{coefficient of } x \text{ on the bottom}}$. Here this would be $y = \frac{4}{3}$.
  - The second is a curve given by a quadratic polynomial divided by a quadratic polynomial, for example, $y = \frac{5x^2+x+5}{2x^2-2x+1}$ as $x \to \pm\infty$, This has a horizontal asymptote at $y = \frac{\text{coefficient of } x^2 \text{ on the top}}{\text{coefficient of } x^2 \text{ on the bottom}}$. Here this would be $y = \frac{5}{2}$.
  - Thirdly, when the curve is given by a linear polynomial divided by a quadratic polynomial, it will generally have the x-axis (y = 0) as a horizontal asymptote.
- That if the degree of the numerator is one greater than the degree of the denominator, then there is an ***oblique*** asymptote, which can be found by dividing out.
- How to sketch graphs of the form $y^2 = f(x)$

Graph Sketching

Rational functions

To sketch the graph of $y = \frac{N(x)}{D(x)}$:

- Find the intercepts – that is where the graph cuts the axes.
- Find any asymptotes – the vertical asymptotes occur at values of x which make the denominator zero.
- Examine the behaviour of the graph near to the vertical asymptotes; a good way to do this is to find out what the value of y is for values of x very close to the vertical asymptote
- Examine the behaviour around any non-vertical asymptotes, i.e. as x tends to $\pm\infty$.

Example Sketch the curve $y = \frac{2x^2+3x+1}{x^2+6x+8}$

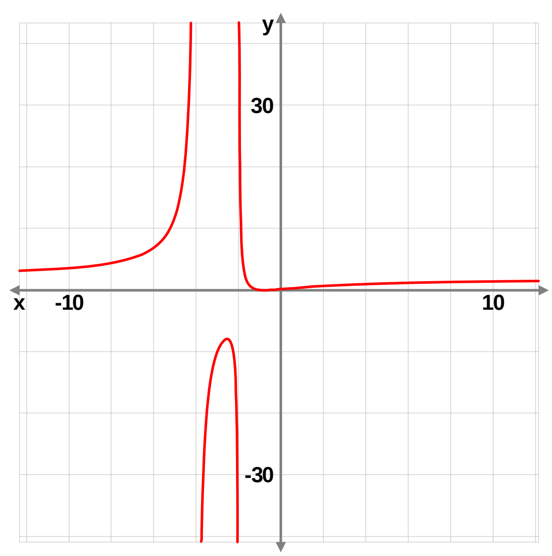
Sketch Solution (remember to label all the key features of your graph when answering exam questions)

The curve can be written as $y = \frac{(2x+1)(x+1)}{(x+2)(x+4)}$.

If $x = 0$ then $y = 0.125$. So the y intercept is (0, 0.125)
Setting $y = 0$ gives, $x = -0.5$ and $x = -1$. So the x intercepts are (-0.5, 0) and (-1, 0).

The denominator is zero when $x = -2$ and when $x = -4$ so these are the vertical asymptotes

Also $y = \frac{2x^2+3x+1}{x^2+6x+8} = \frac{2(x^2+6x+8)-9x-15}{x^2+6x+8} = 2 - \frac{9x+15}{x^2+6x+8}$
so $y = 2$ is a horizontal asymptote.



REVISION SHEET – FP2 (OCR)

THE SOLUTION OF EQUATIONS

The main ideas are:

- Understanding geometrically staircase and cobweb diagrams/sequences and how their convergence relates to gradient considerations.
- The Newton Rapshon Method.

Before the exam you should know:

- How to draw staircase and cobweb diagrams and anticipate the behaviour of sequences from the pictorial representations.
- The relationship between the gradient of F near to the root and the rate of convergence of the sequence to that root.
- The working of the Newton-Raphson Method in geometrical terms.
- And be able to use both methods to approximate roots of equations.

Basics

- It is possible to rearrange *any* equation into the form $x = \mathrm{F}(x)$.
- It may be that the recurrence relation $x_{n+1} = \mathrm{F}(x_n)$, where x_0 is given or chosen appropriately, produces a convergent sequence.
- If this is the case, the limit of the sequence will be a root of $x = \mathrm{F}(x)$ and therefore it will be a root of the original equation.

Example

Consider the equation $x = \sqrt{1+\sin x}$. Use the iteration $x_{r+1} = \sqrt{1+\sin x_r}$ with $x_0 = 1$ to find this root correct to 4 decimal places.

Solution

The iteration $x_{r+1} = \sqrt{1+\sin x_r}$ with $x_0 = 1$, gives

r	**x$_r$**
1	**1**
2	**1.3570081**
3	**1.406141602**
4	**1.409423644**
5	**1.409612592**
6	**1.409623354**
7	**1.409623967**

$1.3570081... = \sqrt{1+\sin(1)}$

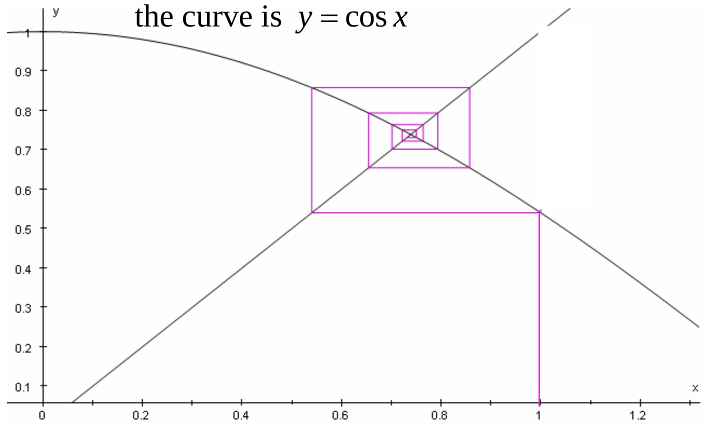
$1.406141602.. = \sqrt{1+\sin(1.3570081....)}$

The last three values calculated are all 1.4096 to 4 decimal places. In fact the root, to 4 decimal places, is 1.0496 as with $\mathrm{f}(x) = x - \sqrt{1+\sin x}$ f(1.40955) = -0.0002… and f(1.40865) = +0.00007…..

Important note: You can get your calculator to perform these calculations very quickly using the 'ANS' feature.

Staircase and Cobweb Diagrams

Example: Approximating a root of $x = \cos x$, the straight line is $y = x$ and the curve is $y = \cos x$



The steps taken to form such a diagram to find an approximation to a root of $x = \mathrm{F}(x)$ are

- Draw a graph with the line $y = x$ and the curve $y = \mathrm{F}(x)$ and mark the point at which they cross whose x-coordinate you are trying to approximate.
- Starting with your initial estimate of x_0
  1. draw the vertical line through $x = x_0$, to meet the curve $y = \mathrm{F}(x)$.
  2. from this point draw a horizontal line to meet $y = x$.
  3. from this point draw a vertical line to meet $y = \mathrm{F}(x)$.
  4. from this point draw a horizontal line to meet $y = x$
  5. and so on….

(Notice how the line $y = x$ is used to transfer the value of x_1 on the y-axis to the x-axis so that it can be "input" into g again to find x_2.)

Rate of Convergence

With experience in drawing staircase and cobweb diagrams you will see how the convergence of the sequence produces depends on the gradient of $\mathrm{F}(x)$ around the root of $x = \mathrm{F}(x)$. In fact if ε_n is the error in x_n as an approximation to the root α (to which the sequence is converging) then:

$$\varepsilon_{n+1} \approx \mathrm{F}'(\alpha)\varepsilon_n \text{ if } \mathrm{F}'(\alpha) \neq 0 \text{ and } \varepsilon_{n+1} \text{ is approximately proportional to } \varepsilon_n{}^2 \text{ if } \mathrm{F}'(\alpha) = 0$$

Newton Raphson-Method

To generate a sequence of values converging to a root of $\mathrm{f}(x) = 0$, near to $x = x_0$, use the following iterative formulae: $x_{r+1} = x_r - \dfrac{\mathrm{f}(x_r)}{\mathrm{f}'(x_r)}$. This method has second order convergence.

Below the Newton Raphson method is being used to approximate a solution of $x^3 + 2x - 1 = 0$. So we have $x_{r+1} = x_r + \dfrac{x_r{}^3 + 2x_r - 1}{3x_r{}^2 + 2}$ in this case. Using a starting value of $x_0 = 1$ gives

$$x_1 = x_0 - \frac{x_0{}^3 + 2x_0 - 1}{3x_0{}^2 + 2} = 1 - \frac{1 + 2 - 1}{3 + 2} = 0.6$$

$$x_2 = x_1 - \frac{x_1{}^3 + 2x_1 - 1}{3x_1{}^2 + 2} = 0.6 - \frac{0.6^3 + 2 \times 0.6 - 1}{3 \times 0.6^2 + 2} = 0.4649350634$$

Then $x_3 = 0.453467173$, $x_4 = 0.453397654$, $x_5 = 0.453397651$, $x_6 = 0.453397651$.

Notice how quickly the sequence converges, this is in fact because the method is an example of an iteration of the form $x_{n+1} = \mathrm{F}(x_n)$ with $\mathrm{F}'(\alpha) = 0$

Important note: Once again, you can get your calculator to perform these calculations very quickly using the 'ANS' feature.