REVISION SHEET – FP2 (MEI)

CALCULUS

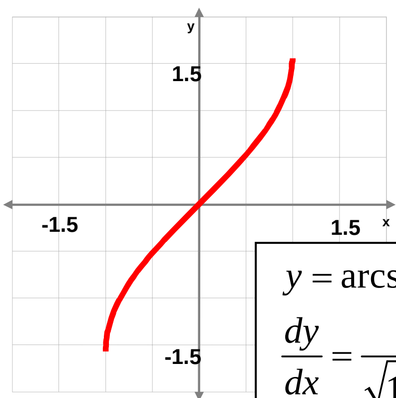
The main ideas are:

- Calculus using inverse trig functions & hyperbolic trig functions and their inverses.
- Maclaurin series

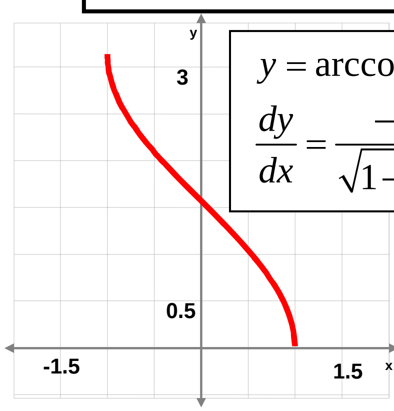
Before the exam you should know:

- That you can differentiate the trig functions, the hyperbolic trig functions and their inverses.
- That you can apply the standard rules for differentiation (product rule, quotient rule and chain rule) to functions which involve the above.
- That you can integrate trig functions and hyperbolic trig functions.
- That you can integrate, arcsin(x), arccos(x), arctan(x), arccot(x), arsinh(x), arcosh(x) etc using integration by parts.
- Your trig identities and hyperbolic function identities and how to use them in integration problems. Particularly get familiar with useful substitutions to make for certain problems.

Differentiating the Inverse Trig Functions

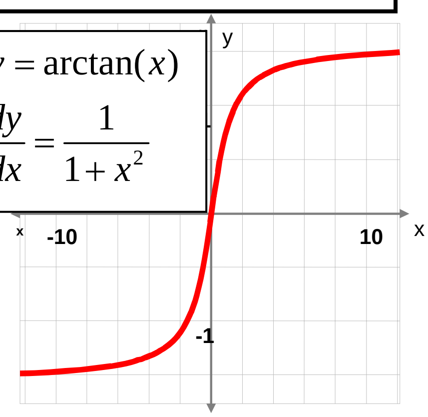


$$y = \arcsin(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$



$$y = \arccos(x) \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$y = \arctan(x) \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$



It is important to be aware of what the range is for each of these, namely:

$$-\frac{\pi}{2} \le \arcsin \le \frac{\pi}{2}, \qquad 0 \le \arccos \le \pi, \qquad -\frac{\pi}{2} \le \arctan \le \frac{\pi}{2}$$

Standard Calculus of Inverse Trig and Hyperbolic Trig Functions

$$y = \arcsin(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$y = \arccos(x) \qquad \frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

$$y = \arctan(x) \qquad \frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y = \operatorname{arsinh}(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$$

$$y = \operatorname{arcosh}(x) \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$$

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2-a^2}} = \operatorname{arcosh}\left(\frac{x}{a}\right) + c$$

$$\int \frac{1}{\sqrt{x^2+a^2}} = \operatorname{arsinh}\left(\frac{x}{a}\right) + c$$

Calculus using these functions

The examples below are very typical and show most of the common tricks. Note – details of all substitutions have been omitted, make sure you understand how to do them in this case and also in the case of a definite integral.

- $\int \frac{1}{\sqrt{4x^2+16x+32}}dx = \frac{1}{2}\int \frac{1}{\sqrt{(x+2)^2+4}}dx = \frac{1}{2}\text{arsinh}\left(\frac{x+2}{2}\right)+c$
- $\int \frac{4}{\sqrt{5+3x-9x^2}}dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{5}{9}-\left(x^2-\frac{x}{3}\right)}}dx = \frac{4}{3}\int \frac{1}{\sqrt{\frac{21}{36}-\left(x-\frac{1}{6}\right)^2}}dx = \frac{4}{3}\arcsin\left(\frac{6\left(x-\frac{1}{6}\right)}{\sqrt{21}}\right)+c = \frac{4}{3}\arcsin\left(\frac{6x-1}{\sqrt{21}}\right)+c$
- $\int \frac{3}{\sqrt{2x^2+4x-10}}dx = \frac{3}{\sqrt{2}}\int \frac{1}{\sqrt{(x+1)^2-6}}dx = \frac{3}{\sqrt{2}}\text{arcosh}\left(\frac{x+1}{\sqrt{6}}\right)+c$
- $y = \text{arcosh}(x^2) \Rightarrow \frac{dy}{dx} = \frac{2x}{\sqrt{x^4-1}}$ (to see this use the chain rule, set $z = x^2$ and then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$).

Some useful integration tricks

Splitting up an integration: e.g. $\int_1^5 \frac{x+5}{x^2+4}dx = \int_1^5 \frac{x}{x^2+4}dx + \int_1^5 \frac{5}{x^2+4}dx$

By inspection: e.g. Since $\ln(x^2+4)$ gives $\frac{2x}{x^2+4}$ when differentiated, we have $\int \frac{x}{x^2+4}dx = \frac{1}{2}\ln(x^2+4)+c$ or since $(x^2+1)^{\frac{1}{2}}$ gives $x(x^2+1)^{-\frac{1}{2}}$ when differentiated, we have $\int \frac{x}{\sqrt{x^2+1}}dx = \sqrt{x^2+1}+c$

Using clever substitutions: e.g. the substitution $u = \sinh(x)$ will help you with $\int \sqrt{x^2+1}dx$.

Maclaurin Series

The Maclaurin expansion for a function f(x) as far as the term in x^n looks as follows.

$$f(x) \approx f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ... + \frac{x^n}{n!}f^{(n)}(0)$$

The Maclaurin ***series*** is obtained by including infinitely many terms (i.e. not terminating the sum as above). It may only be valid for certain values of x. Examples include:

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$ which is valid for all x, $\frac{1}{1-2x} = 1 + 2x + 4x^2 + 8x^3 + ...$ which is valid only when $|x| < \frac{1}{2}$, note that this second example is the same as the binomial expansion of $(1-2x)^{-1}$.

Useful tips

- You can find the Maclaurin series of, e.g. f(2x), by taking the series for f(x) and replacing the x's with 2x.
- If g is the derivative of f then you can find the Maclaurin series for g by differentiating the one for f term by term.
- Likewise, if g is the integral of f then you can find the Maclaurin series for g by integrating the one for f term by term (caution – don't forget the constant of integration, this will be g(0)).

REVISION SHEET – FP2 (MEI)

COMPLEX NUMBERS

The main ideas are:

- De Moivre's Theorem and its applications
- Exponential notation
- Using both of the above to get formulae by summing C+jS series
- n[th] roots of complex numbers

Before the exam you should know:

- How to multiply and divide complex numbers in polar form.
- What de Moivre's theorem is and how to apply it.
- About the exponential notation $e^{j\theta} = \cos\theta + j\sin\theta$, $z = x + yj = re^{j\theta}$
- How to apply de Moivre's theorem to finding multiple angle formulae and to summing series.
- About the n *n*th roots of unity, including how to represent them on an Argand diagram.

De Moivre's Theorem

De Moivre's Theorem states that $(\cos\theta + j\sin\theta)^n = \cos n\theta + j\sin n\theta$ for any integer *n*. Some applications of this are shown below.

Example 1 Evaluate $(1+j)^{12}$.

Solution The first thing to do is to write $1+j$ in polar form. This is just $1+j = \sqrt{2}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$

Therefore
$$\begin{aligned}(1+j)^{12} &= \left(\sqrt{2}\right)^{12}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)^{12} \\ &= 64(\cos 3\pi + j\sin 3\pi) \\ &= 64(\cos\pi + j\sin\pi) \\ &= 64(-1+0) \\ &= -64\end{aligned}$$

Note: in example 2 on the right it is typical to be asked to go on to integrate $\sin^6\theta$. De Moivre's theorem can also be used to express multiple angles in terms of powers of the trig functions in a very straightforward way.

Example 2 Express $\sin^6\theta$ in terms of multiple angles.

Solution If $z = \cos\theta + j\sin\theta$ then $2j\sin\theta = z - z^{-1}$.

So

$$\begin{aligned}(2j)^6\sin^6\theta &= \left(z - z^{-1}\right)^6 \\ &= z^6 - 6z^5z^{-1} + 15z^4z^{-2} - 20z^3z^{-3} + 15z^2z^{-4} - 6zz^{-5} + z^{-6} \\ &= z^6 + z^{-6} - 6\left(z^4 + z^{-4}\right) + 15\left(z^2 + z^{-2}\right) - 20 \\ &= 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\end{aligned}$$

Therefore,

$$-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20$$

$$\begin{aligned}\sin^6\theta &= \frac{20 - 2\cos 6\theta + 12\cos 4\theta - 30\cos 2\theta}{64} \\ &= \frac{10 - \cos 6\theta + 6\cos 4\theta - 15\cos 2\theta}{32}.\end{aligned}$$

Exponential notation for complex numbers

Exponential notation begins with $e^{j\theta} = \cos\theta + j\sin\theta$. This means that any complex number, *z*, can be written in polar form as $z = x + yj = re^{j\theta}$ where *r* is the modulus of z and θ is the argument of *z*.

Example *(of using the exponential notation and De Moivre's theorem to sum C+jS.)*

i. Show that $\left(2+e^{j\theta}\right)\left(2+e^{-j\theta}\right)=5+4\cos\theta$.

ii. Let $S=\frac{\sin\theta}{2}-\frac{\sin 2\theta}{2^2}+\frac{\sin 3\theta}{2^3}-\frac{\sin 4\theta}{2^4}+.....$

By considering $C-jS$ where $C=1-\frac{\cos\theta}{2}+\frac{\cos 2\theta}{2^2}-\frac{\cos 3\theta}{2^3}+\frac{\cos 4\theta}{2^4}-...$ show that $S=\frac{2\sin\theta}{5+4\cos\theta}$.

Solution

$$\begin{aligned}\text{i)}\left(2+e^{j\theta}\right)\left(2+e^{-j\theta}\right)&=4+2e^{j\theta}+2e^{-j\theta}+e^{j\theta}e^{-j\theta}\\&=5+2\left(e^{j\theta}+e^{-j\theta}\right)\\&=5+2\left[\left(\cos\theta+j\sin\theta\right)+\left(\cos(-\theta)+j\sin(-\theta)\right)\right]\\&=5+2\left(\cos\theta+j\sin\theta+\cos\theta-j\sin\theta\right)\\&=5+2\times(2\cos\theta)\\&=5+4\cos\theta.\end{aligned}$$

$$\begin{aligned}\text{ii)}\ C-jS&=1-\frac{\cos\theta}{2}-j\frac{\sin\theta}{2}+\frac{\cos 2\theta}{2^2}+j\frac{\sin 2\theta}{2^2}-\frac{\cos 3\theta}{2^3}-j\frac{\sin 3\theta}{2^3}+...\\&=1-\left(\frac{\cos\theta+j\sin\theta}{2}\right)+\left(\frac{\cos 2\theta+j\sin 2\theta}{2^2}\right)-\left(\frac{\cos 3\theta+j\sin 3\theta}{2^3}\right)+...\\&=1-\left(\frac{\cos\theta+j\sin\theta}{2}\right)+\frac{\left(\cos\theta+j\sin\theta\right)^2}{2^2}-\frac{\left(\cos\theta+j\sin\theta\right)^3}{2^3}+...\\&=1-\left(\frac{\cos\theta+j\sin\theta}{2}\right)+\left(\frac{\cos\theta+j\sin\theta}{2}\right)^2-\left(\frac{\cos\theta+j\sin\theta}{2}\right)^3+...\\&=1-\frac{e^{j\theta}}{2}+\left(\frac{e^{j\theta}}{2}\right)^2-\left(\frac{e^{j\theta}}{2}\right)^3+...\end{aligned}$$

This is just a geometric series with first term 1 and common ratio $-\frac{e^{j\theta}}{2}$. Its sum is given by

$$1-\frac{e^{j\theta}}{2}+\left(\frac{e^{j\theta}}{2}\right)^2-\left(\frac{e^{j\theta}}{2}\right)^3+...=\frac{1}{1-\left(-\frac{e^{j\theta}}{2}\right)}=\frac{2}{2+e^{j\theta}}.$$

Now the real and imaginary parts of $\frac{2}{2+e^{j\theta}}$ need to be calculated. Part i) is useful because it gives

$$C-jS=\frac{2}{2+e^{j\theta}}=\left(\frac{2}{2+e^{j\theta}}\right)\left(\frac{2+e^{-j\theta}}{2+e^{-j\theta}}\right)=\frac{2\left(2+e^{-j\theta}\right)}{5+4\cos\theta}=\frac{2\left(2+\cos\theta-j\sin\theta\right)}{5+4\cos\theta}.$$

Equating imaginary parts gives $S=\frac{2\sin\theta}{5+4\cos\theta}$.

n^th^ roots of complex numbers

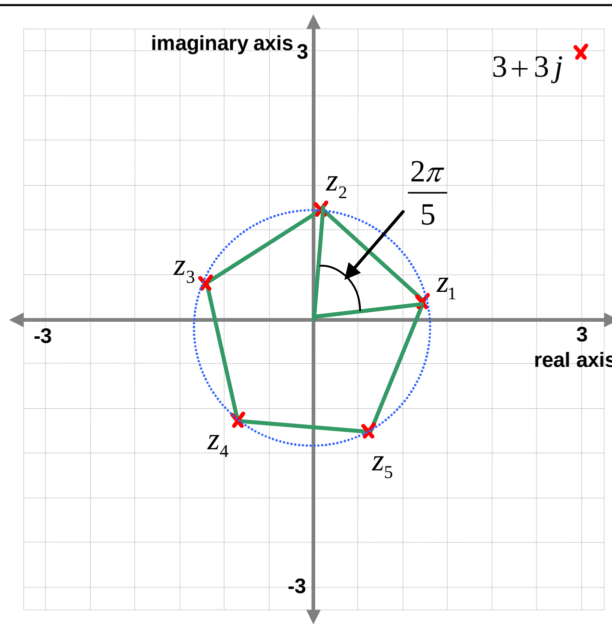
The non-zero complex number $r(\cos\theta+j\sin\theta)$ has n different nth roots, which are:

$$r^{\frac{1}{n}}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+j\sin\left(\frac{\theta+2k\pi}{n}\right)\right),$$

where k = 0, 1, 2, …, $n-1$.

nth roots of complex numbers are best thought about geometrically, the diagram shows the 5th roots of 3+3j.

You should be able to express these roots in polar form using the exponential notation.



REVISION SHEET – FP2 (MEI)

HYPERBOLIC TRIG FUNCTIONS

The main ideas are:

- Definitions of the hyperbolic trig functions and their inverses.
- Working with the hyperbolic trig functions
- Identities involving hyperbolic trig functions

Before the exam you should know:

- The definitions $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- That you can prove that $\text{arccosh}(x) = \ln(x + \sqrt{x^2 - 1})$, $\text{arsinh}(x) = \ln(x + \sqrt{x^2 + 1})$ $\text{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$
- Your trig identities and hyperbolic function identities, experience will tell you when it is best to work in the exponential form when dealing with equations.
- How to prove hyperbolic identities from the definitions $\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$, it's worth practicing indices for this.

The Hyperbolic Trig Functions

These are defined as:

$\sinh(x) = \frac{e^x - e^{-x}}{2}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$, $\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$.

For example, $\sinh(\ln 10) = \frac{e^{\ln 10} - e^{-\ln 10}}{2} = \frac{10 - \frac{1}{10}}{2} = \frac{99}{20}$.

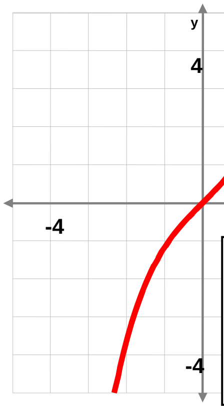
The Inverse Hyperbolic Trig Functions

Just as the hyperbolic trig functions are defined in terms of e^x, their inverses can be expressed in term of logs. In fact $\text{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$, $\text{arsinh}(x) = \ln(x + \sqrt{x^2 + 1})$, $\text{artanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$. You should be able to prove (and use) all of these. Here is the proof that $\text{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$.

Let $y = \text{ar}\cosh(x)$, then $x = \cosh(y) = \frac{e^y + e^{-y}}{2}$. Rearranging this gives $0 = e^y - 2x + e^{-y}$. Multiplying this by e^y gives $0 = e^{2y} - 2xe^y + 1$. This is a quadratic in e^y and using the formula for the roots of a quadratic gives $e^y = \frac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1}$. Taking logs gives $y = \text{ar}\cosh(x) = \ln(x \pm \sqrt{x^2 - 1})$. The value corresponding to the minus sign is rejected here, you should know why.

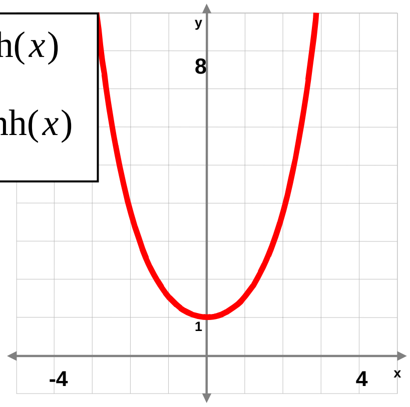
These expressions can be used to give exact values of the inverse hyperbolic trig functions in term of logs. For example, $\text{arcosh}\left(\frac{5}{3}\right) = \ln\left(\frac{5}{3} + \sqrt{\left(\frac{5}{3}\right)^2 - 1}\right) = \ln\left(\frac{5}{3} + \sqrt{\frac{16}{9}}\right) = \ln(3)$.

Graphs of the Hyperbolic Trig Functions



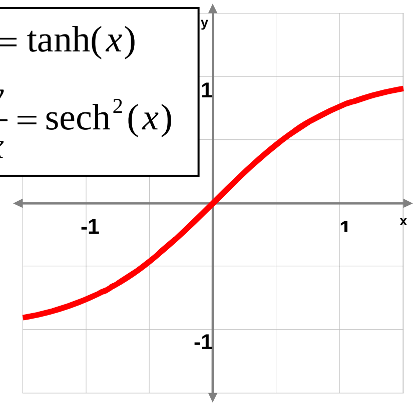
$y = \sinh(x)$

$\frac{dy}{dx} = \cosh(x)$



$y = \cosh(x)$

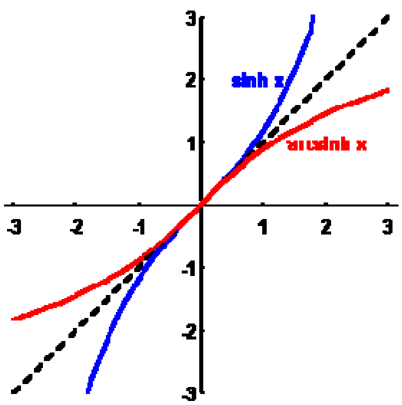
$\frac{dy}{dx} = \sinh(x)$



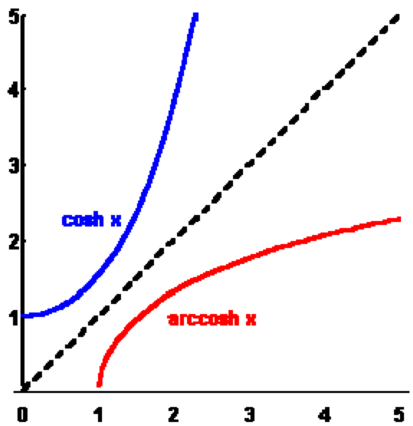
$y = \tanh(x)$

$\frac{dy}{dx} = \text{sech}^2(x)$

Graphs of the Inverse Hyperbolic Trig Functions



You must also know the graphs of the inverse hyperbolic trig functions, arsinh, arcosh and artanh. As for any function these are obtained by reflecting the respective graphs of sinh, cosh and tanh in $y = x$. The examples of arsinh and arcosh are shown here. Notice that arcosh(x) is only defined for x greater than or equal to 1.



Identities Involving Hyperbolic Trig Functions

Identities involving hyperbolic trig functions include:

$$\cosh^2 u - \sinh^2 u = 1, \quad \cosh(2u) = \cosh^2 u + \sinh^2 u, \quad \sin(u+v) = \sinh(u)\cosh(v) + \cosh(u)\sinh(v)$$

The only difference between a hyperbolic trig identity and the corresponding standard trig identity is that the sign is reversed when a product of two sines is replaced by a product of two sinhs. This is called Osborn's Rule.

You can prove any hyperbolic trig identity using their definitions and should be able to do this for the exam.

Equations Involving Hyperbolic Trig Functions

Example Solve the equation $13\cosh x + 5\sinh x = 20$ giving your answer in terms of natural logarithms.

Solution $13\cosh x + 5\sinh x = 20 \Rightarrow 13\left(\frac{e^x + e^{-x}}{2}\right) + 5\left(\frac{e^x - e^{-x}}{2}\right) = 20$

$$\Rightarrow 18e^x + 8e^{-x} - 40 = 0$$

$$\Rightarrow 9e^{2x} - 20e^x + 4 = 0 \Rightarrow (9e^x - 2)(e^x - 2) = 0$$

$$\Rightarrow e^x = \frac{2}{9} \text{ or } e^x = 2$$

$$\Rightarrow x = \ln\left(\frac{2}{9}\right) \text{or } x = \ln 2$$

REVISION SHEET – FP2 (MEI)

INVESTIGATION OF CURVES

The main ideas in this topic are:

- The cartesian, parametric and polar forms of equations
- Symmetry and periodicity of curves, asymptotes, nodes and loops
- Calculus techniques for curves in Cartesian, parametric and polar form
- Conic sections

Before the exam you should know:

- How to use your graphic calculator efficiently.
- How to convert between different forms of the equations.
- The links between the equation of a curve and its shape.
- Using calculus for curves given in Cartesian and parametric and polar form and understanding what they will show.
- The standard equations of conics in Cartesian and parametric form.

Defining a curve

Types of equation:

- Cartesian - Parametric - Polar

To convert from parametric to Cartesian, you need to eliminate the parameter.
It may be possible to obtain a simple relationship between the parameter, x and y. This can then be substituted into the equation for x or y. If the parametric form involves trig functions, you may be able to use identities like $\sin^2\theta + \cos^2\theta = 1$ and $\tan^2\theta + 1 = \sec^2\theta$
To convert between polar and Cartesian form, use the relationships $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$

Example: convert the parametric equations $x = 4\sec t$, $y = 5\tan t$ into (a) Cartesian form (b) polar form

(a) $x^2 = 16\sec^2 t$ $\qquad$ $y^2 = 25\tan^2 t$

$x^2 = 16\sec^2 t$ $\qquad$ $\frac{y^2}{25} = \tan^2 t$

$\frac{x^2}{16} = 1 + \tan^2 t$

$\frac{x^2}{16} = 1 + \frac{y^2}{25} \Rightarrow \frac{x^2}{16} - \frac{y^2}{25} = 1$

(b) using $x^2 + y^2 = r^2$ gives $16\sec^2 t + \tan^2 t = r^2$

$16(1 + \tan^2 t) + \tan^2 t = r^2$

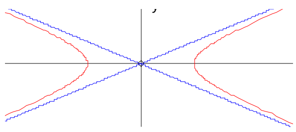
$41\tan^2 t + 16 = r^2$

Features of curves

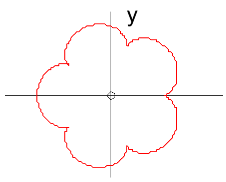
The important features of curves to recognise are

- Symmetry and periodicity.
- Vertical, horizontal and oblique asymptotes.
- Cusps, loops and dimples.
- Nodes (points where a curve crosses over itself).
- Nodes (points where a curve crosses over itself).

The **Hyperbola:** $x = a\sec t$, $y = b\tan t$ has oblique asymptotes.



Epicycloid: x = kacost – a cos kt, y= kasint - asinkt



This example has K = 6.
It has 5 dimples.

Symmetry and asymptotes

Vertical asymptotes are the values of x which make the denominator zero when the equation is in Cartesian form.
Horizontal and oblique asymptotes depend on the behaviour of the curve as $x \to \pm\infty$. The clue is in the orders of the numerator and denominator of the graph:

- If the order of the denominator is greater than the order of the numerator, then $y \to 0$ as $x \to \pm\infty$, and so the x axis is a horizontal asymptote.
- If the order of the denominator is equal to the order of the numerator, then $y \to k$ for some constant k as, $x \to \pm\infty$ and the line $y = k$ is a horizontal asymptote.
- If the order of the denominator is less than the order of the numerator, then y numerically increases without limit as $x \to \pm\infty$ and there is an oblique asymptote.

Using calculus

For curves given in **cartesian** and **parametric** form calculus techniques are used to find

- the equations of tangents and normals;
- the maximum and minimum values of x and y
- the maximum and minimum distances of a curve from the origin.

For curves given in **polar** form calculus techniques are used to find

- the maximum and minimum distance of the curve from the pole;
- the points on the curve where the tangent is parallel or perpendicular to the initial line.

Example: determine the equations of the vertical and oblique asymptotes of the curve $y = \dfrac{3x^2 + 7x + 4}{3x + 1}$

Solution: Vertical asymptote $3x + 1 = 0 \Rightarrow x = ⅓$

Oblique asymptote $(Ax + B)(3x + 1) + C = 3x^2 + 7x + 4 \Rightarrow 2Ax^2 + (2B - A)x + (C - B) = 2x^2 + 4x + 3$

Equating coefficients gives A = 1, B = 2, C=2. Hence $y = x + 2 + \dfrac{2}{3x+1}$ so the oblique asymptote is the line $y = x + 2$

Conic sections

The conics can be thought of as the curves given by slicing through a double cone. They can be defined in terms of the locus of a point P in relation to a fixed point S (the focus) and a fixed line *d* (the directrix). A conic is the locus of P such that the ratio of distance of P from S and the distance of P from *d* is a constant *e* (the eccentricity).

	Parabola e=1	Ellipse 0<e<1	Hyperbola e>1	Rectangular hyperbola e=√2
Cartesian form	$y^2 = 4ax$	$\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$	$\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$	$xy = c^2$
Parametric form	$x = 2at^2$ $y = 2at$	$x = a\cos t$ $y = b\sin t$	$x = a\sec t$ $y = b\tan t$	$x = ct$ $y = \dfrac{c}{t}$

Example: A curve has parametric equations $x = \dfrac{2t}{1+t^2}$, $y = \dfrac{t^2}{1+t^2}$.

(i) Use a graphical calculator to draw the curve for $-10 \le t \le 0$

(ii) The curve is part of a conic. Name the conic.

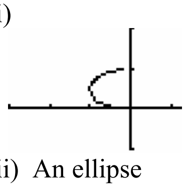
(iii) Write down the co-ordinates of the centre point of the conic and of the points where it crosses the *y*-axis.

(iv) Show that for the curve S, $\dfrac{dy}{dx} = \dfrac{t}{1-t^2}$.

(v) Find the values of *t* at the points where the curve is parallel to the *y*-axis, and the Cartesian coordinates of these points.

Solution:

i)



ii) An ellipse

iii) Centre point (0, ½); the curve cuts the y axis at (0, 0) and (1, 0)

iv) $\dfrac{dx}{dt} = \dfrac{2(1+t^2) - 2t.2t}{(1+t^2)^2} = \dfrac{2t - 2t^2}{(1+t^2)^2}$; $\dfrac{dy}{dt} = \dfrac{2t(1+t^2) - t^2.2t}{(1+t^2)^2} = \dfrac{2t}{(1+t^2)^2} \Rightarrow \dfrac{dy}{dx} = \dfrac{\frac{2t}{(1+t^2)^2}}{\frac{2-2t^2}{(1+t^2)^2}} = \dfrac{t}{1-t^2}$

v) $\dfrac{dy}{dx} = \infty$ when $1 - t^2 = 0 \Rightarrow t = \pm 1$.

When t = 1, x = 1, y = ½. When t = -1, x = -1, y = ½. Co-ordinates are (1, ½) and (-1, ½)

REVISION SHEET – FP2 (MEI)

MATRICES

The main ideas are:

- The inverse of a 3 by 3 matrix
- Solving 3 simultaneous equations in 3 unknowns, and interpreting the solutions geometrically
- Finding eigenvalues and eigenvectors
- Diagonalisation and applications

Before the exam you should know:

- How to find the determinant of a 3×3 matrix and how to invert a 3×3 matrix whose determinant is not zero.
- The language of matrices – singular, non-singular, cofactor, adjugate.
- How to solve 3 simultaneous equations in 3 unknowns. What to expect when the underlying coefficient matrix is zero. How to interpret each case geometrically.
- What is meant by an eigenvalue and an eigenvector of a 3×3 matrix and how to find them.
- How to diagonalise a matrix M where possible. In other words how to find a matrix S such that $S^{-1}MS$ is a diagonal matrix.
- Applications of diagonalising matrices, such as finding large powers of a matrix.
- The Cayley Hamilton Theorem and its applications.

The Determinant of a 3×3 matrix

If $M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ then

$\det M = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$. This is "expanding by the first row", it is possible to expand by any row or column and you should know how to do this.

The Inverse of a 3×3 matrix

If $M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$ then $M^{-1} = \frac{1}{\det M}\begin{pmatrix} A_1 & -B_1 & C_1 \\ -A_2 & B_2 & -C_2 \\ A_3 & -B_3 & C_3 \end{pmatrix}$,

where A_1 for example is the determinant of the 2×2 matrix which is left after removing the column and row containing a_1 from M, B_3 is the determinant of the 2×2 matrix which is left after removing the column and row containing b_3 from M *etc.*

Matrices and Simultaneous Equations

The three simultaneous equations in three variables

$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

are equivalent to the matrix equation $M\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$ where $M = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$.

If det M $\neq$ 0, M is non-singular, M^{-1} exists and the equations have a unique solution.
If det M = 0, M is singular, M^{-1} does not exist and either:

(i) the equations are inconsistent and have no solutions
(ii) the equations are consistent and have infinitely many solutions.

The three equations may be regarded as the equations of three planes in three-dimensional place and you should be able to interpret the solutions in terms of the configurations of these planes.

Eigenvalues and Eigenvectors

For a given matrix M, an eigenvector is a vector which is mapped to a multiple of itself by M. For example since

$$\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix} = 5\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ with eigenvalue 5.

To find eigenvectors of a matrix M
(i) Form the characteristic equation: det(M - λI) = 0
(ii) Solve the characteristic equation to find the eigenvalues, λ
(iii) For each eigenvalue λ find an eigenvector v $\neq 0$ by solving Mv = λv

Diagonalisation

A diagonal matrix is a square matrix in which all the entries off the top left to bottom right diagonal are zero. If M is any matrix and S is a matrix whose columns are eigenvectors of M then, if S is invertible, $S^{-1}MS$ is a diagonal matrix.

Suppose $S^{-1}MS = D$ where D is a diagonal matrix. Then $D^n = (S^{-1}MS)^n = (S^{-1}MS)(S^{-1}MS)\dots(S^{-1}MS) = S^{-1}M^nS$.
This gives $M^n = SD^nS^{-1}$ which is an easy way to calculate high powers of M.

The Cayley Hamilton Theorem

The Cayley Hamilton Theorem says that a matrix *satisfies* its own characteristic equation. The following example shows what this means.

Suppose $M = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$. The characteristic equation of M is $\begin{vmatrix} 1-\lambda & 2 \\ 1 & 4-\lambda \end{vmatrix} = 0$ which is $\lambda^2 - 5\lambda + 2 = 0$.

Now we substitute λ = M into the characteristic equation. This gives (notice how 2 becomes 2I below).

$M^2 - 5M + 2I = 0$ i.e. $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}^2 - 5\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 10 \\ 5 & 18 \end{pmatrix} - \begin{pmatrix} 5 & 10 \\ 5 & 20 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

The Cayley Hamilton Theorem can be used to calculate high powers of a matrix quickly. In the above example we have $M^2 = 5M - 2I$. This gives

$$\begin{aligned} M^4 &= (5M - 2I)(5M - 2I) = 25M^2 - 20M + 4I \\ &= 25(5M - 2I) - 20M + 4I \\ &= 105M - 46I \end{aligned}$$

REVISION SHEET – FP2 (MEI)

POLAR COORDINATES

The main ideas are:

- What Polar Coordinates are
- Conversion between Cartesian and Polar Coordinates
- Curves defined using Polar Coordinates
- Calculating areas for curves defined using Polar Coordinates

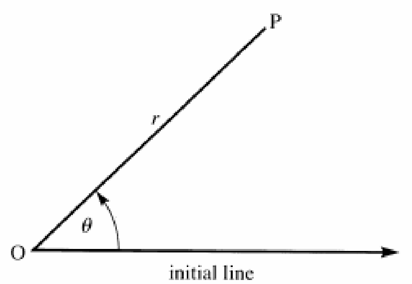
Before the exam you should know:

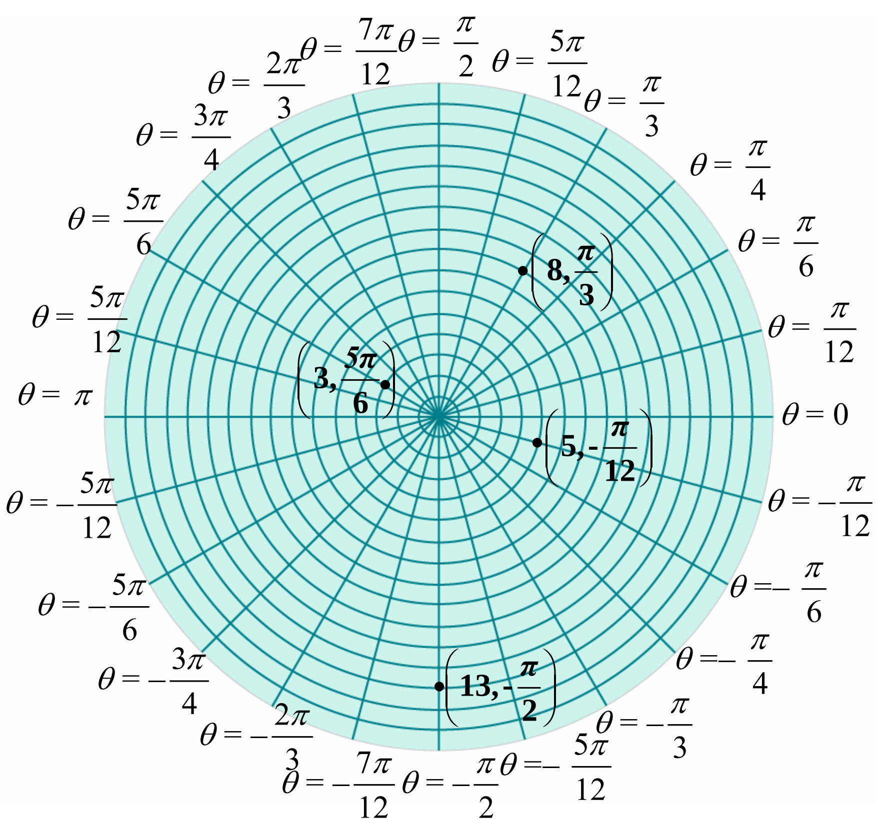
- How to change between polar coordinates (r, θ) and Cartesian coordinates (x, y) use $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$.
- You'll need to be very familiar with the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$ and be able to give exact values of the trig functions for multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$.
- How to sketch a curve given by a polar equation.
- The area of a sector is given by $\int_\alpha^\beta \frac{1}{2}r^2\,d\theta$.

How Polar Coordinates Work

You will be familiar with using Cartesian Coordinates (x, y) to specify the position of a point in the plane. Polar coordinates use the idea of describing the position of a point P by giving its distance r from the origin and the angle θ between OP and the positive x-axis. The angle θ is positive in the anticlockwise sense from the initial line. If it is necessary to specify the polar coordinates of a point uniquely then you use those for which $r > 0$ and $-\pi < \theta \le \pi$.

It is sometimes convenient to let r take negative values with the natural interpretation that $(-r, \theta)$ is the same as $(r, \theta + \pi)$.





It is easy to change between polar coordinates (r, θ) and Cartesian coordinates (x, y) since $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2+y^2}$ and $\tan\theta = \frac{y}{x}$. You need to be careful to choose the right quadrant when finding θ, since the equation $\tan\theta = \frac{y}{x}$ always gives two values of θ, differing by π. Always draw a sketch to check which one of these is correct.

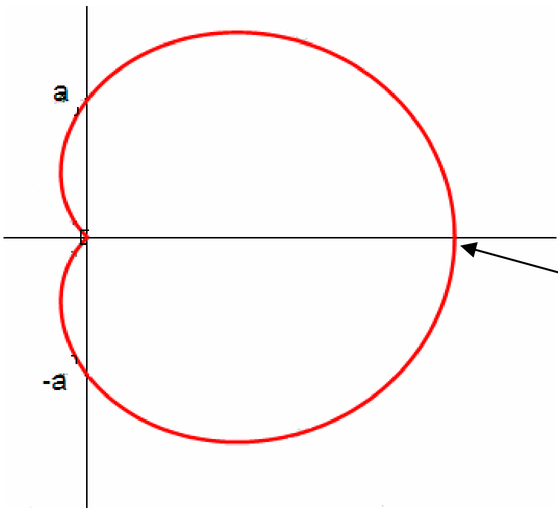
The Polar Equation of a Curve

The points (r, θ) for which the values of r and θ are linked by a function f form a curve whose polar equation is $r = f(\theta)$. A good way to draw a sketch of a curve is to calculate r for a variety of values of θ.

Example Sketch the curve which has polar equation $r = a(1+\sqrt{2}\cos\theta)$ for $-\frac{3}{4}\pi \le \theta \le \frac{3}{4}\pi$, where a is a positive constant.

Solution Begin by calculating the value or r for various values of θ. This is shown in the table. The curve can now be sketched.

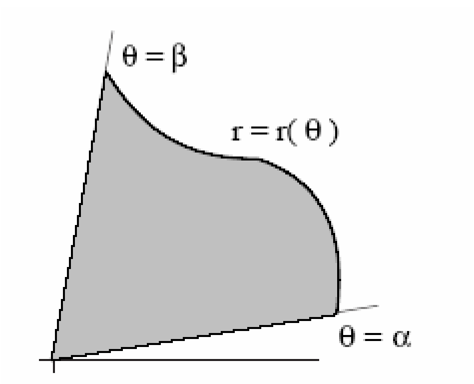
θ	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$
r	0	a	$2a$	$a\left(1+\sqrt{2}\right)$	$2a$	a	0



It's a good exercise to try to spot the points given in the table above in polar coordinates on the curve shown here.

For example the point $\left(a\left(1+\sqrt{2}\right),0\right)$ is here.

The Area of a Sector



The area of the sector shown in the diagram is $\int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta$

Example

A curve has polar equation $r = a(1+\sqrt{2}\cos\theta)$ for $-\frac{3}{4}\pi \le \theta \le \frac{3}{4}\pi$, where a is a positive constant. Find the area of the region enclosed by the curve.

Solution The area is clearly twice the area of the sector given by $0 \le \theta \le \frac{3}{4}\pi$. Therefore the area is

$$2\int_0^{\frac{3\pi}{4}}\frac{1}{2}r^2\,d\theta = a^2\int_0^{\frac{3\pi}{4}}\left(1+\sqrt{2}\cos\theta\right)^2 d\theta = a^2\int_0^{\frac{3\pi}{4}}\left(1+2\sqrt{2}\cos\theta+2\cos^2\theta\right)d\theta$$

$$= a^2\int_0^{\frac{3\pi}{4}}\left(2+2\sqrt{2}\cos\theta+\cos 2\theta\right)d\theta$$

$$= a^2\left[2\theta+2\sqrt{2}\sin\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{3\pi}{4}}$$

$$= \frac{3}{2}(\pi+1)a^2$$

Note Even though r can be negative for certain values of θ, $\frac{1}{2}r^2$ is always positive, so there is no problem of 'negative areas' as there is with curves below the x-axis in cartesian coordinates.

Be careful however when considering loops contained inside loops.