MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

FP2, Further Methods for Advanced Mathematics
(Version B: reference to new book)

Topic 1: Calculus

Topic 2: Polar Coordinates

Topic 3: Complex Numbers

Topic 4: Power Series

Topic 5: Matrices

Option 1: Hyperbolic Functions

Option 2: Investigation of curves

MEI, Oak House, 9 Epsom Centre, White Horse Business Park, Trowbridge, Wiltshire. BA14 0XG.
Company No. 3265490 England and Wales Registered with the Charity Commission, number 1058911
Tel: 01225 776776. Fax: 01225 775755.

References:
Chapter 1
Pages 1-11

Exercise 1B
Q. 2, 4, 6(ii), 7(ii)

Inverse Trigonometrical Functions

y = arcsin$x$ is the inverse function of y = sin$x$.

$$\frac{\mathrm{d}(\arcsin x)}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}(\arccos x)}{\mathrm{d}x}=-\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}(\arctan x)}{\mathrm{d}x}=\frac{1}{1+x^2}$$

References:
Chapter 1
Pages 11-14

Example 1.6
Page 12

Exercise 1C
Q. 1(i),(ii), 2(i),(ii)

Integration involving Inverse Functions

$$\frac{\mathrm{d}(\arcsin x)}{\mathrm{d}x}=\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow\int\frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x=\arcsin x+c$$

$$\frac{\mathrm{d}(\arctan x)}{\mathrm{d}x}=\frac{1}{1+x^2}$$

$$\Rightarrow\int\frac{1}{1+x^2}\,\mathrm{d}x=\arctan x+c$$

This can be seen by making the substitution

$$x=\tan\theta\Rightarrow\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$$

and $1+x^2=1+\tan^2\theta=\sec^2\theta$

$$\Rightarrow\int\frac{1}{1+x^2}\,\mathrm{d}x=\int\frac{\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta$$

$$=\int\mathrm{d}\theta=\theta+c=\arctan x+c$$

When making a substitution to complete a definite integral, either convert the limits to the values of the function being used or turn your integrand back into a function of x and then substitute the limits.

References:
Chapter 1
Pages 15-17

Example 1.9
Page 15

Exercise 1D
Q. 1(i), 4(i),(v), 6(i)

Harder integrals

If the function in the denominator is of the form ax^2+bx+c then completing the square allows the procedure above to be used.

E.g. $x^2+4x+7\equiv(x+2)^2+3$

So $I=\int\frac{1}{x^2+4x+7}\,\mathrm{d}x=\int\frac{1}{(x+2)^2+3}\,\mathrm{d}x$

Substitute $(x+2)=\sqrt{3}\tan\theta$

$\Rightarrow(x+2)^2+3=3\tan^2\theta+3=3\sec^2\theta$

and $\mathrm{d}x=\sqrt{3}\sec^2\theta\,\mathrm{d}\theta$

$$\Rightarrow I=\int\frac{1}{(x+2)^2+3}\,\mathrm{d}x=\int\frac{\sqrt{3}\sec^2\theta\,\mathrm{d}\theta}{3\sec^2\theta}=\frac{1}{3}\sqrt{3}\theta+c$$

$$=\frac{1}{\sqrt{3}}\arctan\left(\frac{x+2}{\sqrt{3}}\right)+c$$

E.g. Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin 2x)$.

$u=2x\Rightarrow\frac{\mathrm{d}u}{\mathrm{d}x}=2$

$$\Rightarrow\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin 2x)=\frac{\mathrm{d}}{\mathrm{d}u}(\arcsin u).\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$=\frac{1}{\sqrt{1-u^2}}.2=\frac{2}{\sqrt{1-4x^2}}$$

E.g. Find $\int_0^{\frac{1}{2}}\frac{1}{1+4x^2}\,\mathrm{d}x$.

Note first that $\frac{\mathrm{d}(\arctan x)}{\mathrm{d}x}=\frac{1}{1+x^2}\Rightarrow\int\frac{1}{1+x^2}\,\mathrm{d}x=\arctan x+c$

Substitute $2x=\tan\theta$

$\Rightarrow 2\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sec^2\theta$ and $1+4x^2=1+\tan^2\theta=\sec^2\theta$

When $x=0,\tan\theta=0\Rightarrow\theta=0$

When $x=\frac{1}{2},\tan\theta=1\Rightarrow\theta=\frac{\pi}{4}$

$$\Rightarrow\int_0^{\frac{1}{2}}\frac{1}{1+4x^2}\,\mathrm{d}x=\int_0^{\frac{\pi}{4}}\frac{\frac{1}{2}\sec^2\theta}{\sec^2\theta}\,\mathrm{d}\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\mathrm{d}\theta$$

$$=\frac{1}{2}[\theta]_0^{\frac{\pi}{4}}=\frac{\pi}{8}$$

E.g. Find $\int\frac{x(x-2)}{(x+1)(x^2+2)}\,\mathrm{d}x$

$$\frac{x(x-2)}{(x+1)(x^2+2)}\equiv\frac{1}{x+1}-\frac{2}{x^2+2}$$

(This is found by partial fractions, covered in C4.)

$$\Rightarrow\int\frac{x(x-2)}{(x+1)(x^2+1)}\,\mathrm{d}x=\int\left(\frac{1}{x+1}-\frac{2}{x^2+2}\right)\mathrm{d}x$$

$$=\int\frac{1}{x+1}\mathrm{d}x-2\int\frac{1}{x^2+2}\,\mathrm{d}x=\ln|x+1|-2.\frac{1}{\sqrt{2}}\arctan\frac{x}{\sqrt{2}}+c$$

E.g. Find $\int_1^2\frac{1}{\sqrt{1+2x-x^2}}\,\mathrm{d}x$.

$1+2x-x^2\equiv 2-(x-1)^2$

So let $(x-1)=\sqrt{2}u$

Then $1+2x-x^2\equiv 2-(x-1)^2=2-2u^2$

and $\mathrm{d}x=\sqrt{2}\,\mathrm{d}u$

When $x=1,u=0$ and $x=2,u=\frac{1}{\sqrt{2}}$

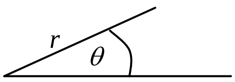
$$\Rightarrow\int_1^2\frac{1}{\sqrt{1+2x-x^2}}\,\mathrm{d}x=\int_0^{\frac{1}{\sqrt{2}}}\frac{\sqrt{2}}{\sqrt{2(1-u^2)}}\,\mathrm{d}u$$

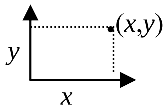
$$=\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{\sqrt{(1-u^2)}}\,\mathrm{d}u=[\arcsin u]_0^{\frac{1}{\sqrt{2}}}=\frac{\pi}{4}$$

References: Chapter 2 Page 20

Cartesian coordinates identify a point by an ordered pair (x, y) of distances from two, usually perpendicular, axes which intersect at the **origin**, O.
Polar coordinates identify a point by an ordered pair, (r, θ) where r is the distance from a fixed point, O, called the **pole**, and θ is the angle turned through in an anticlockwise direction from a fixed line through O, called the **initial line.**
The point is uniquely defined providing r and θ are defined such that $r \geq 0$ and $0 \leq \theta < 2\pi$.
(Angles are usually expressed in radians.)

Exercise 2A Q. 2

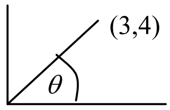




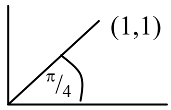
E.g. The point with Cartesian coordinates (3, 4) has polar coordinates (r,θ) where

$$\theta = \tan^{-1}\frac{4}{3} = 0.927 \text{ radians}$$

and $r = \sqrt{3^2 + 4^2} = 5$



The point with cartesian coordinates (1, 1) has polar coordinates $(\sqrt{2}, \pi/4)$.



References: Chapter 2 Page 21

Conversion between Polars and Cartesians

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$x^2 + y^2 = r^2 \Rightarrow r = \sqrt{x^2 + y^2}$$

$$\tan\theta = \frac{y}{x}$$

References: Chapter 2 Pages 23-26

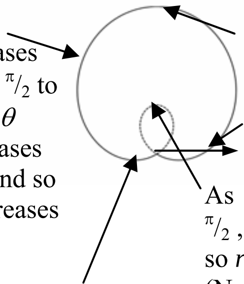
Example 2.1 Page 23

Polar Equations of Curves
The polar equation of a curve can be expressed in the form $r = f(\theta)$.

Curves may be sketched by plotting specific points or by considering the value of r over a range of values of θ.

Exercise 2B Q. 1, 2

E.g. Sketch the curve $r = 1 + 2\sin\theta$.



As θ increases from $^{\pi}/_{2}$ to π, $\sin\theta$ decreases to 0 and so r decreases to 1

As θ increases from 0 to $^{\pi}/_{2}$, $\sin\theta$ increases to 1 and so r increases to 3

When $\theta = 0$, $r = 1$.

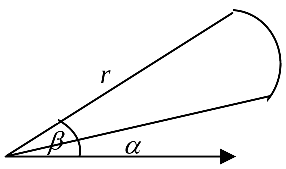
As θ increases from π to $^{3\pi}/_{2}$, $\sin\theta$ decreases to -1 and so r decreases to -1.
(Note that there is a point here when $r = 0$.
This is when $\sin\theta = -^{1}/_{2}$, i.e. $\theta = ^{7\pi}/_{6}$.)

As θ increases from $^{3\pi}/_{2}$ to 2π, $\sin\theta$ increases from -1 to 0 and so r increases to 1, once again through 0, when $\theta = ^{11\pi}/_{6}$.

References: Chapter 2 Pages 27-28

Area of Sector

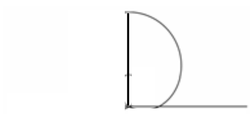
$$\text{Area} = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2 \, d\theta$$



Exercise 2C Q. 2

E.g. Find the area of the sector of the curve $r = 1 + 2\sin\theta$ from $\theta = 0$ to $^{\pi}/_{2}$.

$$\text{Area} = \frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2 \, d\theta$$



$$= \frac{1}{2}\int_0^{\pi/2} (1 + 2\sin\theta)^2 \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 + 4\sin\theta + 4\sin^2\theta\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \left(1 + 4\sin\theta + 2\left(1 - \cos 2\theta\right)\right) d\theta$$

$$= \frac{1}{2}\left[3\theta - 4\cos\theta - \sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2}\left(\frac{3\pi}{2} - 4 \times 0 - 0\right) - \frac{1}{2}(0 - 4 - 0)$$

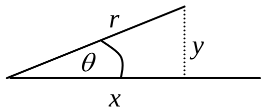
$$= \frac{3\pi}{4} + 2$$

Summary FP2 Topic 3: **Complex Numbers – 1**

References: Chapter 3 Pages 32-35

Exercise 3A Q. 3, 15, 18

The Polar form of a complex number $x + y\text{j}$ is given as (r, θ) where r is the modulus of the complex number and θ is the anticlockwise angle turned through from the positive x (or real) axis.



$\tan\theta = \dfrac{y}{x}$

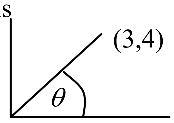
$r = \sqrt{x^2 + y^2}$

To enable this representation to be unique, we define the range of θ to be $-\pi < \theta \leq \pi$, where the measurement is usually in radians.

E.g. The point (3 + 4j) with Cartesian coordinates (3, 4) has Polar coordinates (r, θ) where

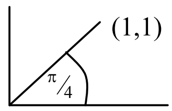
$\theta = \tan^{-1}\dfrac{4}{3} = 0.927$ radians

$r = \sqrt{3^2 + 4^2} = 5$



The point 1 +j has polar coordinates $\left(\sqrt{2}, \dfrac{\pi}{4}\right)$.

Note the comparison between the polar form of complex numbers and polar coordinates (Topic 2.)



References: Chapter 3 Page 36

Exercise 3B Q. 2, 8, 10

Sets of points

The equation $\arg(z - p) = k\pi$ is a half line with constant angle $k\pi$ from the point p. The other half represents the equation $\arg(z - p) = (k-1)\pi$.

References: Chapter 3 Pages 37-38

Exercise 3C Q. 4, 6, 14

Multiplication and division in polar form.

If $z_1 = (r_1, \theta_1) = r_1(\cos\theta_1 + \text{j}\sin\theta_1)$

and $z_2 = (r_2, \theta_2) = r_2(\cos\theta_2 + \text{j}\sin\theta_2)$

then $z_1 z_2 = (r_1 r_2, \theta_1 + \theta_2)$

$= r_1 r_2(\cos(\theta_1 + \theta_2) + \text{j}\sin(\theta_1 + \theta_2))$

and $\dfrac{z_1}{z_2} = \left(\dfrac{r_1}{r_2}, \theta_1 - \theta_2\right) = \dfrac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + \text{j}\sin(\theta_1 - \theta_2))$

It can also be seen that if $z_1 = (r_1, \theta_1)$ and $z_2 = (r_2, \theta_2)$

then $|z_1 z_2| = |z_1||z_2|$

and $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

E.g. $z_1 = \left(3, \dfrac{\pi}{4}\right)$, $z_2 = \left(2, \dfrac{\pi}{3}\right)$

$z_1 z_2 = \left(6, \dfrac{7\pi}{12}\right)$, $\dfrac{z_1}{z_2} = \left(1.5, -\dfrac{\pi}{12}\right)$

References: Chapter 3 Pages 40-41

Exercise 3D Q. 1(i), 2(i), 3(i)

de Moivre's Theorem

If $z = (r, \theta) = r(\cos\theta + \text{j}\sin\theta)$

and n is any integer

$z^n = (r^n, n\theta) = r^n(\cos n\theta + \text{j}\sin n\theta)$

E.g. If $z = \left(\cos\dfrac{\pi}{4} + \text{j}\sin\dfrac{\pi}{4}\right)$

then $z^8 = \left(\cos\dfrac{\pi}{4} + \text{j}\sin\dfrac{\pi}{4}\right)^8$

$= (\cos 2\pi + \text{j}\sin 2\pi) = 1$

Note that the equivalent algebraic form of z is

$z = \dfrac{1}{\sqrt{2}}(1 + \text{j}) \Rightarrow z^8 = \left(\dfrac{1}{\sqrt{2}}\right)^8 (1 + \text{j})^8$

$= \dfrac{1}{16}(1 + 8\text{j} - 28 - 56\text{j} + 70 + 56\text{j} - 28 - 8\text{j} + 1)$

$= \dfrac{1}{16}(1 - 28 + 70 - 28 + 1) = \dfrac{16}{16} = 1$

References: Chapter 3 Pages 42-43

Example 3.2 Page 41

Exercise 3E Q. 3,5(i)

Multiple angles using de Moivre's Theorem

If $z = (\cos\theta + \text{j}\sin\theta)$

and n is any integer

$z^n = (\cos\theta + \text{j}\sin\theta)^n = (\cos n\theta + \text{j}\sin n\theta)$

The powered bracket should be expanded using the binomial theorem (and using $\text{j}^2 = -1$) and then equate real and imaginary parts.

E.g. If $z = (\cos\theta + \text{j}\sin\theta)$

$z^3 = (\cos\theta + \text{j}\sin\theta)^3 = (\cos 3\theta + \text{j}\sin 3\theta)$

$= \cos^3\theta + 3\text{j}\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - \text{j}\sin^3\theta$

So $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

$= \cos^3\theta - 3\cos\theta(1 - \cos^2\theta)$

$= 4\cos^3\theta - 3\cos\theta$

and $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1 - \sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

FP2; Further Methods for Advanced Mathematics
Version B: page 4
Competence statements j1, 2, 3, 4

References:
Chapter 3
Pages 45-47

Example 3.6
Page 47

Exercise 3F
Q. 1(ii), 4, 6

Complex Exponents

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This comes from the comparison of the infinite series expansions for $\cos\theta, \sin\theta$ and $e^{j\theta}$

i.e. $\cos\theta + j\sin\theta$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - ...\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...\right)$$

$$= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} + ...$$

$$= 1 + (j\theta) + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + ...$$

$$= e^{j\theta}$$

In particular: $e^{j\pi} = \cos\pi + j\sin\pi = -1$

E.g. Express $1 - e^{j\theta}$ in the form $a\sin\frac{\theta}{2}.e^{jk\theta}$

$$e^{j\theta} = \cos\theta + j\sin\theta \Rightarrow 1 - e^{j\theta} = 1 - \cos\theta - j\sin\theta$$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2} \Rightarrow 1 - \cos\theta = 2\sin^2\frac{\theta}{2}$$

$$1 - e^{j\theta} = 2\sin^2\frac{\theta}{2} - 2j\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - j\cos\frac{\theta}{2}\right)$$

$$= -2j^2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - j\cos\frac{\theta}{2}\right) = -2j\sin\frac{\theta}{2}\left(j\sin\frac{\theta}{2} - j^2\cos\frac{\theta}{2}\right)$$

$$= -2j\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2} - j\sin\frac{\theta}{2}\right) = -2j\sin\frac{\theta}{2}e^{\frac{1}{2}j\theta}$$

References:
Chapter 3
Page 49

Summations using complex numbers

Series expansions involving $\cos\theta$ or $\sin\theta$ may be done using complex numbers, de Moivre's theorem and equating real and imaginary parts.

E.g. Find the sum of the series $\sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + ...$

Let $S = \sin\theta + \frac{1}{2}\sin 2\theta + \frac{1}{4}\sin 3\theta + ...$

and $C = \cos\theta + \frac{1}{2}\cos 2\theta + \frac{1}{4}\cos 3\theta + ...$

$$\Rightarrow C + jS = (\cos\theta + j\sin\theta) + \frac{1}{2}(\cos 2\theta + j\sin 2\theta) + ...$$

$$= e^{j\theta} + \frac{1}{2}e^{2j\theta} + \frac{1}{4}e^{3j\theta} + ... = e^{j\theta}\left(1 + \frac{1}{2}e^{j\theta} + \frac{1}{4}e^{2j\theta} + ...\right)$$

$$= e^{j\theta}\left(1 + \frac{1}{2}e^{j\theta} + \left(\frac{1}{2}e^{j\theta}\right)^2 + ...\right) = e^{j\theta}\left(1 - \frac{1}{2}e^{j\theta}\right)^{-1}$$

$$= \frac{e^{j\theta}}{\left(1 - \frac{1}{2}e^{j\theta}\right)} = \frac{e^{j\theta}\left(1 - \frac{1}{2}e^{-j\theta}\right)}{\left(1 - \frac{1}{2}e^{j\theta}\right)\left(1 - \frac{1}{2}e^{-j\theta}\right)} = \frac{e^{j\theta} - \frac{1}{2}}{1 - \frac{1}{2}\left(e^{j\theta} + e^{-j\theta}\right) + \frac{1}{4}}$$

$$= \frac{\cos\theta - \frac{1}{2} + j\sin\theta}{\frac{5}{4} - \cos\theta} \Rightarrow S = \frac{4\sin\theta}{5 - 4\cos\theta}$$

Exercise 3G
Q. 1, 4

References:
Chapter 3
Pages 51-56

Example 3.8
Page 53

Complex roots

If $z = r(\cos\theta + j\sin\theta)$,

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\theta + 2k\pi}{n} + j\sin\frac{\theta + 2k\pi}{n}\right)$$

(For the range of the root to be $[0, 2\pi]$, the range of the number must be $[0, 2n\pi]$).

For $k = 0, 1, 2.....(n-1)$, these angles are distinct, giving the n nth roots of z.

Since they all have the same modulus, they all lie on the circle $|z| = \sqrt[n]{r}$ and so they form, on an Argand diagram, a regular n-gon.

Exercise 3H
Q. 2, 6

Exercise 3I
Q. 2, 4

The sum of all *n*th roots of a complex number is 0.

Method 1.
Consider the complex number z_1. The nth roots are roots of the equation $z^n = z_1$.
The sum of roots of this equation is the coefficient of the z^{n-1} term which is zero.

Method 2.
If the n roots are α, β, γ...then these roots form the vertices of a regular n-gon. Adding complex numbers on the Argand diagram is done by drawing them tracking round a polygon.
In this case the numbers being added track round to the starting point. The "resultant" is therefore zero.

Exercise 3I
Q. 9

Exercise 3J
Q. 4

E.g. Find all 3 cube roots of 8.

Write $z^3 = 8$ in polar form $\equiv (8, 0)$

$$\Rightarrow \sqrt[3]{8} = \sqrt[3]{8}\left(\cos\frac{0 + 2k\pi}{3} + j\sin\frac{0 + 2k\pi}{3}\right)$$

$$= (2, 0), \left(2, \frac{2\pi}{3}\right), \left(2, \frac{4\pi}{3}\right)$$

$$= 2, 2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right), 2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

Note that the sum is

$$2 + 2\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) + 2\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

$$2 - 1 + j\sqrt{3} - 1 - j\sqrt{3} = 0$$

Note also that the product

$$= 8\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) = 8\left(\frac{1}{4} + \frac{3}{4}\right) = 8$$

References:
Chapter 4
Pages 69-76

Example 4.1
Page 72

Exercise 4A
Q. 1(i), 9

Maclaurin's Expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + ...$$

providing that f(x) and all its derivatives exist at $x = 0$

If the series with n terms tends to a limit as n tends to infinity, then we say that the series converges as n tends to infinity and it converges to f(x).

E.g. Find a series expansion for $y = \frac{1}{(1+x)^2}$

$f(x) = (1+x)^{-2};\ f(0) = 1$

$f'(x) = -2(1+x)^{-3};\ f'(0) = -2$

$f''(x) = -2\times-3(1+x)^{-4};\ f''(0) = 6$

$\Rightarrow f(x) = 1 - 2x + 2\times3\times\frac{x^2}{2} - 2\times3\times4\times\frac{x^3}{3!} + ...$

$\Rightarrow f(x) = 1 - 2x + 3x^2 - 4x^3 + ...$

E.g. Find a series expansion for $y = \cos x$

$f(x) = \cos x;\quad f(0) = 1$

$f'(x) = -\sin x;\quad f'(0) = 0$

$f''(x) = -\cos x;\quad f''(0) = -1$

$f'''(x) = \sin x;\quad f'''(0) = 0$

$f''''(x) = \cos x;\quad f''''(0) = 1$

$\Rightarrow f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - .....$

It can be seen that odd powers have coefficient 0 and even powers have coefficients alternating 1 and -1.

$\Rightarrow (r+1)$th term is $\frac{(-1)^r x^{2r}}{(2r)!}$

References:
Chapter 4
Page 76

Series expansions for standard functions

$\Rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^r}{r!} + ...$

Valid for all values of x.

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ... + \frac{(-1)^{r+1}x^r}{r} + ...$

Valid for $-1 < x \le 1$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ..... + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + ...$

Valid for all x.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - ..... + \frac{(-1)^r x^{2r}}{(2r)!} + ...$

Valid for all x.

$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - ..... + \frac{(-1)^r x^{2r+1}}{(2r+1)} + ...$

Valid for $|x| \le 1$

E.g. Find a series expansion for $y = \arcsin x$ up to the term in x^2, and hence find an approximation to $\int_{0.1}^{0.2} \arcsin x\, dx$.

$f(x) = \arcsin x;\quad f(0) = 0$

$f'(x) = \frac{1}{\sqrt{1-x^2}};\ f'(0) = 1$

$f''(x) = \frac{x}{(1-x^2)^{3/2}};\ f''(0) = 0$

$\Rightarrow f(x) = x + ...$

$\Rightarrow \int_{0.1}^{0.2} \arcsin x\, dx \approx \int_{0.1}^{0.2} x\, dx = \left[\frac{x^2}{2}\right]_{0.1}^{0.2}$

$= 0.02 - 0.005 = 0.015$

References:
Chapter 4
Page 78

Example 4.3
Page 78

Exercise 4B
Q. 1(i), 3

An alternative approach

Using the function notation, f(x) and the first derivative, f '(x) with the associated values f(0) and f '(0), etc, then sometimes it is possible to obtain a relationship between derivatives.

E.g. $f''(x) = af'(x) + bf(x)$.

Then $f''(0) = af'(0) + bf(0)$

and $f'''(x) = af''(x) + bf'(x)$, etc.

This relationship may be continued indefinitely.

E.g. Find a series expansion for $f(x) = e^x \cos x$.

$f(0) = 1$

$f'(x) = e^x\cos x - e^x \sin x;\ f'(0) = 1$

$f''(x) = e^x\cos x - e^x\sin x - e^x\sin x - e^x\cos x = -2e^x\sin x$

$\Rightarrow f''(x) = 2f'(x) - 2f(x)$

$\Rightarrow f''(0) = 2f'(0) - 2f(0) = 2 - 2 = 0$

$\Rightarrow f'''(x) = 2f''(x) - 2f'(x)$

$\Rightarrow f'''(0) = 2f''(0) - 2f'(0) = -2$

$\Rightarrow f''''(x) = 2f'''(x) - 2f''(x) \Rightarrow f''''(0) = -4$

$\Rightarrow f(x) = e^x\cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} + ...$

References: Chapter 5 Pages 84-86

Example 5.1 Page 86

Exercise 5A Q. 1(i),2(i), 5

Determinants

For the determinant $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ the **minor** of the element a_1, A_1, is the 2×2 determinant obtained by eliminating the row and column containing a_1.

Then $\Delta = a_1A_1 + a_2A_2 + a_3A_3$

where $A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $A_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, $A_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

E.g. Find the value of the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 7 & 8 \\ 3 & 10 & 15 \end{vmatrix}.$$

$$\Delta = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 7 & 8 \\ 1 & 3 & 7 \end{vmatrix} \quad (\text{Row 3} - \text{Row 2})$$

$$= \begin{vmatrix} 1 & 2 & 5 \\ 0 & 3 & -2 \\ 0 & 1 & 2 \end{vmatrix} \quad \begin{pmatrix} \text{Row 3} - \text{Row 1} \\ \text{and Row 2} - 2\times\text{Row 1} \end{pmatrix}$$

$$= \begin{vmatrix} 3 & -2 \\ 1 & 2 \end{vmatrix} = 6-(-2) = 8$$

References: Chapter 5 Pages 87-91

Exercise 5B Q. 1, 12

Rules for calculating determinants

(i) Interchanging two columns (or rows) changes the sign of the determinant. However, cyclic interchange leaves the sign unaltered.

(ii) The value of a determinant is unchanged by subtracting one row from another row (or one column from another column).

(iii) A determinant with a row or column of zeros is zero. From (ii) above, the value of a determinant with identical rows (or columns) is zero.

E.g. Find the value of the determinant

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 6 & 7 \\ 3 & 11 & 13 \end{vmatrix}.$$

$$\Delta = \begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & 5 \\ 3 & 11 & 10 \end{vmatrix} \quad (\text{Column 3} - \text{Column 1})$$

$$= 5\begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & 1 \\ 3 & 11 & 2 \end{vmatrix} \quad \begin{pmatrix} \text{Factor of 5} \\ \text{from Column 3} \end{pmatrix}$$

$$= 5\begin{vmatrix} 1 & 2 & 0 \\ 2 & 6 & 1 \\ -1 & -1 & 0 \end{vmatrix} \quad (\text{Row 3} - 2\times\text{Row2})$$

$$= -5\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = -5\big((-1)-(-2)\big) = -5$$

References: Chapter 5 Pages 93-96

Example 5.3 Page 94

Exercise 5C Q. 1(i), 3, 5

The inverse of a 3 × 3 matrix

Given $\mathrm{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$, $\mathrm{M}^{-1} = \frac{1}{\Delta}\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

where Δ is the value of the determinant.

From above, $\Delta = a_1A_1 + a_2A_2 + a_3A_3$

Note that $a_1B_1 + a_2B_2 + a_3B_3 = 0$

i.e. multiplying out the "wrong" column with a row gives 0.

E.g. Find M^{-1} where $\mathrm{M} = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 7 & 8 \\ 3 & 10 & 15 \end{pmatrix}$.

From above, $\Delta = 8$.

$A_1 = \begin{vmatrix} 7 & 8 \\ 10 & 15 \end{vmatrix} = 25$, $A_2 = -\begin{vmatrix} 2 & 5 \\ 10 & 15 \end{vmatrix} = 20$, $A_3 = \begin{vmatrix} 2 & 5 \\ 7 & 8 \end{vmatrix} = -19$

$B_1 = -\begin{vmatrix} 2 & 8 \\ 3 & 15 \end{vmatrix} = -6$, $B_2 = \begin{vmatrix} 1 & 5 \\ 3 & 15 \end{vmatrix} = 0$, $B_3 = -\begin{vmatrix} 1 & 5 \\ 2 & 8 \end{vmatrix} = 2$

$C_1 = \begin{vmatrix} 2 & 7 \\ 3 & 10 \end{vmatrix} = -1$, $C_2 = -\begin{vmatrix} 1 & 2 \\ 3 & 10 \end{vmatrix} = -4$, $C_3 = \begin{vmatrix} 1 & 2 \\ 2 & 7 \end{vmatrix} = 3$

$$\Rightarrow \mathrm{M}^{-1} = \frac{1}{8}\begin{pmatrix} 25 & 20 & -19 \\ -6 & 0 & 2 \\ -1 & -4 & 3 \end{pmatrix}$$

References: Chapter 5 Pages 98-101

Exercise 5D Q. 3, 10, 16

Simultaneous Equations

2 simultaneous equations in two unknowns or three equations in three unknowns may be written in matrix form, **MX = A**

Then the equations may be solved, since

$\mathbf{X} = \mathbf{M}^{-1}\mathbf{A}$.

This represents the solution provided $\mathbf{M}^{-1}$ exists. If $\mathbf{M}^{-1}$ does not exist then the equations are either inconsistent or the solution is not unique.

Determine whether the following three equations are consistent or inconsistent.

$\pi_1 : 3x + 2y + z - 4 = 0$

$\pi_2 : x + y + 2z - 6 = 0$

$\pi_3 : 3x + y - 4z - 8 = 0$

Det(M) = 0 so no unique solution.

$2\pi_1 - 3\pi_2 \equiv 3x + y - 4z + 10 = 0 \neq \pi_3$ So inconsistent.

References:
Chapter 5
Pages 104-110

Example 5.5
Page 107

Exercise 5E
Q. 1(i),2(i), 6

Eigenvectors and Eigenvalues

If **s** is a non-zero vector such that M**s** = λ**s** for a scalar number, λ, then **s** is called an Eigenvector of M. λ is called an Eigenvalue of M.

If M is a 2 × 2 matrix then there are two Eigenvectors; if M is a 3 × 3 matrix then there are three.

To find the Eigenvalues and Eigenvectors, solve M**s** = λ**s**
i.e. (M – λI)**s** = 0.
As **s** is non-zero, this means that
Det(M – λI) = 0

E.g. $M = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$. $|M-\lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 8 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)^2 - 16 = 0 \Rightarrow 1-\lambda = \pm 4 \Rightarrow \lambda = -3, 5$

Let $\mathbf{s} = \begin{pmatrix} x \\ y \end{pmatrix}$. For $\lambda = 5$, $\begin{pmatrix} 1-5 & 2 \\ 8 & 1-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow -4x + 2y = 0 \Rightarrow y = 2x \Rightarrow$ Eigenvector $= \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

For $\lambda = -3$, $\begin{pmatrix} 1+3 & 2 \\ 8 & 1+3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow 4x + 2y = 0 \Rightarrow y = -2x \Rightarrow$ Eigenvector $= \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

References:
Chapter 5
Pages 113-114

Exercise 5F
Q. 1(i), 2

The diagonal form and powers of M

If **M** is a 2 × 2 matrix with Eigenvectors $\mathbf{s}_1$ and $\mathbf{s}_2$ with associated Eigenvalues λ_1 and λ_2 then the matrix **S** = ($\mathbf{s}_1$, $\mathbf{s}_2$) and **Λ** which is a matrix where the elements of the leading diagonal are the associated Eigenvalues with zeros elsewhere are such that **MS** = **SΛ**.

$MS = S\Lambda \Rightarrow S^{-1}MS = \Lambda$

$\Rightarrow \left(S^{-1}MS\right)^2 = \Lambda^2$

$\Rightarrow S^{-1}MS\,S^{-1}MS = S^{-1}MMS = S^{-1}M^2S = \Lambda^2$

$\Rightarrow M^2 = S\Lambda^2S^{-1}$

Similarly, $M^n = S\Lambda^nS^{-1}$

The similar property is true for a 3 × 3 matrix.

E.g.For M above, $M = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}$.

$\mathbf{s}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\lambda_1 = 5$, $\mathbf{s}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\lambda_2 = -3$

$\Rightarrow S = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \Lambda = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$

Check: $MS = \begin{pmatrix} 1 & 2 \\ 8 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 10 & 6 \end{pmatrix}$

$S\Lambda = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 5 & -3 \\ 10 & 6 \end{pmatrix}$

E.g.For $M = \begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix}$, express M in the form $S\Lambda S^{-1}$ and hence find M^3.

$\begin{vmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (3-\lambda)(1-\lambda) - 8 = 0$

$\Rightarrow \lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda-5)(\lambda+1) = 0 \Rightarrow \lambda = 5, -1$

For $\lambda = 5$, $\begin{pmatrix} 3-5 & 2 \\ 4 & 1-5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow x = y \Rightarrow \mathbf{s}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

For $\lambda = -1$, $\begin{pmatrix} 3+1 & 2 \\ 4 & 1+1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2x + y = 0 \Rightarrow \mathbf{s}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\Rightarrow S = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}, \Lambda = \begin{pmatrix} 5 & 0 \\ 0 & -1 \end{pmatrix}, S^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}, \Lambda^3 = \begin{pmatrix} 5^3 & 0 \\ 0 & -1 \end{pmatrix}$

$\Rightarrow M^3 = \frac{1}{3}\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 125 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 125 & -1 \\ 125 & 2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 249 & 126 \\ 252 & 123 \end{pmatrix} = \begin{pmatrix} 83 & 42 \\ 84 & 41 \end{pmatrix}$

References:
Chapter 5
Pages 114-116

Exercise 5F
Q. 3(ii), 7

The Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation.
For the 2 × 2 matrix M, if the characteristic equation is $\lambda^2 + a\lambda + b = 0$
Then $M^2 + aM + bI = 0$

It follows, for instance, by multiplying through-out by M, that $M^3 + aM^2 + bM = 0$.

This gives an alternative method to find powers of M.

E.g.For M above the characteristic equation is

$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow M^2 - 4M - 5I = 0$

$\Rightarrow M^2 = 4M + 5I = 4\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} + 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 17 & 8 \\ 16 & 9 \end{pmatrix}$

and $M^2 = 4M + 5I \Rightarrow M^3 = 4M^2 + 5M$

$= 4\begin{pmatrix} 17 & 8 \\ 16 & 9 \end{pmatrix} + 5\begin{pmatrix} 3 & 2 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 83 & 42 \\ 84 & 41 \end{pmatrix}$

Summary FP2 Option 1: ***Hyperbolic Functions***

References:
Chapter 6
Pages 123-128

Exercise 6A
Q. 1, 3(i), 7(ii), 9(ii)

Hyperbolic Functions

$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right), \quad \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right)$

$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x} - 1}{e^{2x} + 1}$

$\frac{d(\cosh x)}{dx} = \sinh x, \quad \frac{d(\sinh x)}{dx} = \cosh x$

$\frac{d}{dx}(\tanh x) = \text{sech}^2 x$

E.g. Show that $\sinh 2x = 2\sinh x \cosh x$ and find an expression for $\cosh 2x$.
From definitions,

$2\sinh x \cosh x = 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$

$= \frac{1}{2}\left(e^{2x} - e^{-2x}\right) = \sinh 2x$

$\cosh 2x = \frac{1}{2}\left(e^{2x} + e^{-2x}\right) = \frac{1}{2}\left(\left(e^x + e^{-x}\right)^2 - 2\right)$

$= \frac{1}{2}\left(e^x + e^{-x}\right)^2 - 1 = 2\cosh^2 x - 1$

References:
Chapter 6
Page 125

Example 6.1
Page 126

Osborne's Rule
Hyperbolic identities are identical to the trigonometrical identities except that whenever there is a product (or implied product) of two sinhs the sign is reversed.

E.g. $\cos^2 x + \sin^2 x = 1$
and $\cosh^2 x - \sinh^2 x = 1$

E.g. $\cos 2x = 1 - 2\sin^2 x$
and $\cosh 2x = 1 + 2\sinh^2 x$
but $\cos 2x = 2\cos^2 x - 1$
and $\cosh 2x = 2\cosh^2 x - 1$

Exercise 6B
Q. 2

Compound Angle formulae

$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$

$\cosh(x-y) = \cosh x \cosh y - \sinh x \sinh y$

$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

$\sinh(x-y) = \sinh x \cosh y - \cosh x \sinh y$

$\tanh(x+y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$\tanh(x-y) = \frac{\tanh x - \tanh y}{1 - \tanh x \tanh y}$

E.g. Prove the compound angle formula for $\tanh(x+y)$ and find an expression for $\tanh 2x$.

$\tanh(x+y) = \frac{\sinh(x+y)}{\cosh(x+y)}$

$= \frac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y}$

$= \frac{\frac{\sinh x \cosh y}{\cosh x \cosh y} + \frac{\cosh x \sinh y}{\cosh x \cosh y}}{\frac{\cosh x \cosh y}{\cosh x \cosh y} + \frac{\sinh x \sinh y}{\cosh x \cosh y}}$

$= \frac{\frac{\sinh x}{\cosh x} + \frac{\sinh y}{\cosh y}}{1 + \frac{\sinh x \sinh y}{\cosh x \cosh y}} = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$

$\Rightarrow \tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$

References:
Chapter 6
Page 128

Example 6.2
Page 133

Exercise 6B
Q. 2

Other hyperbolic functions

$\coth x = \frac{1}{\tanh x}, \quad \text{sech} x = \frac{1}{\cosh x}, \quad \text{cosech} x = \frac{1}{\sinh x}$

References:
Chapter 6
Pages 130-134

Exercise 6C
Q. 4(i), 5(i), 6 (i),(ii), 7

Inverse hyperbolic functions

$\text{artanh} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),$

$\text{arcosh} x = \ln\left(x + \sqrt{x^2 - 1}\right)$

$\text{arsinh}\, x = \ln\left(x + \sqrt{x^2 + 1}\right)$

$\frac{d}{dx}(\text{arcosh} x) = \frac{1}{\sqrt{x^2 - 1}}$

$\frac{d}{dx}(\text{arsinh} x) = \frac{1}{\sqrt{x^2 + 1}}$

$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \text{arsinh}\frac{x}{a} + c$

$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \text{arcosh}\frac{x}{a} + c$

E.g. Find $\int_1^2 \frac{1}{\sqrt{x^2 + 2x + 10}} dx$

$x^2 + 2x + 10 = (x+1)^2 + 9$

$\Rightarrow \int_1^2 \frac{1}{\sqrt{x^2 + 2x + 10}} dx = \int_1^2 \frac{1}{\sqrt{(x+1)^2 + 9}} dx$

$= \left[\text{arsinh}\left(\frac{x+1}{3}\right)\right]_1^2 = \text{arsinh} 1 - \text{arsinh}\frac{2}{3}$

$= \ln\left(1 + \sqrt{2}\right) - \ln\left(\frac{2}{3} + \sqrt{\frac{13}{9}}\right) = \ln\left(\frac{3\left(1 + \sqrt{2}\right)}{2 + \sqrt{13}}\right)$

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Competence statements a4, 5, 6, 7, 8

References:
Chapter 7
Pages 138-142

The **locus** of a point is the path traced out by the point as it moves according to a given rule.
There are three ways to describe the locus :

Cartesian equation
A relationship between the x and y coordinates of the point. $f(x,y) = 0$

Parametric equation
The coordinates, x and y are related via a parameter. $x = f(t)$, $y = g(t)$.

Polar equation
Each point in the plane is described in terms of the distance from an origin (called the Pole) and the angle turned through anticlockwise from a fixed line through the pole. $r = f(\theta)$.

References:
Chapter 7
Pages 142-145

Exercise 7A
Q. 1(i), 2, 8

Conversion between forms
Polar - Cartesian and Cartesian - Polar

Use $r^2 = x^2 + y^2$; $x = r\cos\theta, y = r\sin\theta$

or $\cos\theta = \dfrac{x}{r}, \sin\theta = \dfrac{y}{r}$

Parametric - Cartesian
Eliminate the parameter from the equations giving the relationship between x and y.
Parametric—Polar
First convert to Cartesian.

References:
Chapter 7
Pages 148-155

Example 7.2
Page 156

Loops and cusps
A loop is a part of the curve that traces out one area by passing through a given point twice.
A cusp is a point on a curve where two arcs of the curve meet with coincident tangents.

References:
Chapter 7
Pages 155-157

Symmetry and nodes
A point where a curve crosses itself is called a **Node.**
If two values of the parameter of a parametric equation give the same point, then that point is a node.

References:
Chapter 7
Pages 158-159

Example 7.3
Page 158

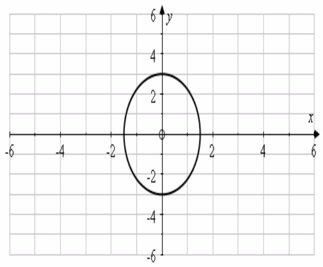
Exercise 7A
Q. 3(i), (ii)

Asymptotes
Horizontal and vertical asymptotes were introduced in FP1. Some curves also have oblique asymptotes.
If the equation of a curve can be rewritten in the form $y = ax + b + f(x)$ where $f(x)$ tends to zero as x tends to infinity then the line $y = ax + b$ is an oblique asymptote.

The curve with equation $4x^2 + y^2 = 9$ is an ellipse.

The parametric equations are:

$x = \dfrac{3}{2}\cos T, \ y = 3\sin T$



giving the polar equation $r = \dfrac{3}{\sqrt{3\cos^2\theta + 1}}$

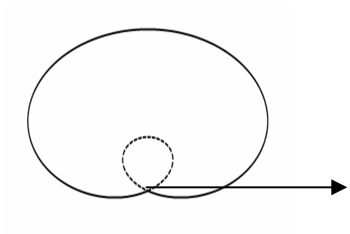
E.g. Convert the polar equation $r = \dfrac{3}{\sqrt{3\cos^2\theta + 1}}$ to cartesian form.

$r = \sqrt{x^2 + y^2}$ and $\cos\theta = \dfrac{x}{r} \Rightarrow x^2 + y^2 = \dfrac{9}{\dfrac{3x^2}{x^2+y^2} + 1}$

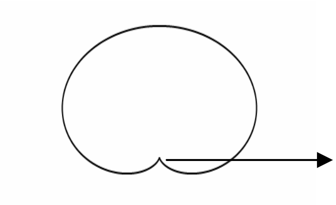
$\Rightarrow x^2 + y^2 = \dfrac{9(x^2+y^2)}{3x^2 + x^2 + y^2} \Rightarrow 1 = \dfrac{9}{4x^2 + y^2}$

$\Rightarrow 4x^2 + y^2 = 9$

E.g. $r = 1 + 2\sin\theta$ contains a loop.



$r = 1 + \sin\theta$ has a cusp.

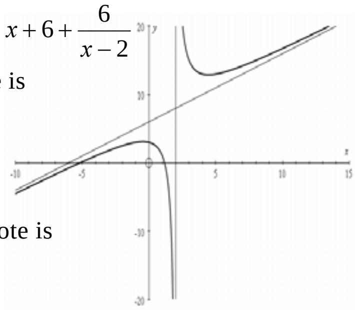


E.g. Find the equations of the asymptotes of the curve $y = 1 + \dfrac{x^2 + 3x - 4}{x - 2}$ and draw the graph.

The curve can be rewritten

$y = 1 + x + 5 + \dfrac{6}{x-2} = x + 6 + \dfrac{6}{x-2}$

$\Rightarrow$ vertical asymptote is
$x = 2$



and oblique asymptote is
$y = x + 6$

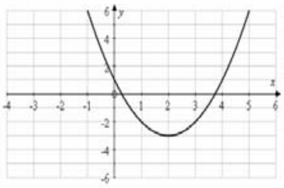
References:
Chapter 7
Pages 160-162

Exercise 7B
Q. 2

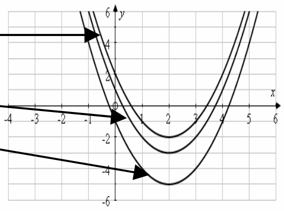
Families of curves

Curves with a common property are called a family of curves.

E.g. $y = x^2 - 4x + 1$ is a curve known as a parabola.



For different values of a, $y = x^2 - 4x + a$ is a family of curves.



References:
Chapter 7
Pages 172-177

Exercise 7C
Q. 6

Using Calculus

Calculus can be used when the curve is given in any of the three forms :

- To find equations of tangents and normals
- To determine maximum and minimum points
- To find maximum and minimum distances from the origin.

References:
Chapter 7
Pages 184-195

Example 7.5
Page 191

Exercise 7D
Q. 3, 6, 7

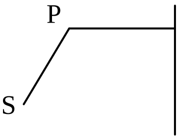
Exercise 7E
Q. 6

Conics

If S is a fixed point and d a fixed line, then the locus of a point P which moves so that the ratio of the distance to the point and to the line is constant is a conic.

The value of the ratio, e, is called the *eccentricity.*

$$|PS| = e|PM|$$



$e = 1$ gives a parabola
$0 < e < 1$ gives an ellipse
$e > 1$ gives a hyperbola

In their simplest forms the cartesian equations of the conics are:

Parabola: $y^2 = 4ax$

Ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

When $a = b$ the ellipse becomes a circle: $x^2 + y^2 = a^2$

Hyperbola: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

When $a = b$ the hyperbola is rectangular: $x^2 - y^2 = a^2$

this can be rewritten $XY = c^2$

In their simplest forms the parametric equations of the conics are:

Parabola: $x = at^2,\ y = 2at$

Ellipse: $x = a\cos t,\ y = b\sin t$

When $a = b$ the ellipse becomes a circle: $x = a\cos t,\ y = a\sin t$

Hyperbola: $x = a\sec t,\ y = b\tan t$

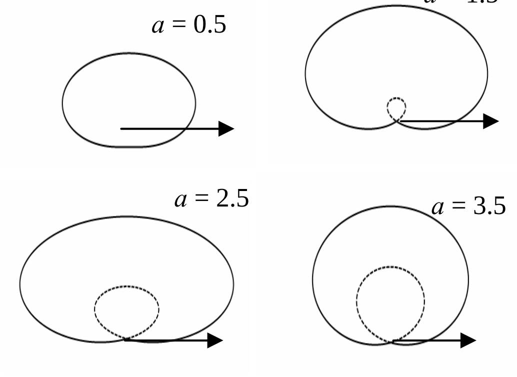
When $a = b$ the hyperbola is rectangular and can be transformed to

$x = ct,\ y = \frac{c}{t}$

E.g. Investigate the family of curves with polar equation $r = 1 + a\sin\theta$ for different values of a.

Two curves are shown on the previous page with $a = 1$ and $a = 2$.
$a = 0$ gives a circle.

The curves shown are for $a = 0.5$, 1.5, 2.5 and 3.5.



E.g. The tangent at the point P $(ap^2, 2ap)$ on the parabola $x = at^2$, $y = 2at$ meets the x-axis at T and the normal at P meets the x-axis at N. Find the area of the triangle PNT.

At any point, $\frac{dx}{dt} = 2at,\ \frac{dy}{dt} = 2a \Rightarrow \frac{dy}{dx} = \frac{1}{t}$

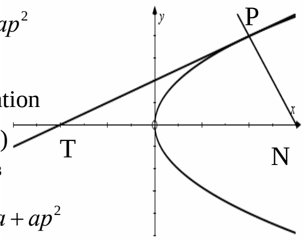
At P the gradient of the tangent is $\frac{1}{p}$

$\Rightarrow$ Tangent has equation $y - 2ap = \frac{1}{p}(x - ap^2)$

$\Rightarrow py = x + ap^2$

$\Rightarrow$ When $y = 0, x = -ap^2$

$\Rightarrow T(-ap^2, 0)$



Normal at P has equation

$y - 2ap = -p(x - ap^2)$

$\Rightarrow y + px = 2ap + ap^3$

$\Rightarrow$ When $y = 0, x = 2a + ap^2$

$\Rightarrow N(2a + ap^2, 0)$

In triangle TPN, length of base = TN

$= 2a + 2ap^2$

Height = y coordinate of P = $2ap$

$\Rightarrow$ Area = $\frac{1}{2}\left(2a + 2ap^2\right)2ap = 2a^2p\left(1 + p^2\right)$