MEI

Mathematics in Education and Industry

MEI Structured Mathematics

Module Summary Sheets

FP3, Further Applications of Advanced Mathematics
(Version B: reference to new book)

Option 1: Vectors

Option 2: Multivariable Calculus

Option 3: Differential Geometry

Option 4: Groups

Option 5: Markov Chains

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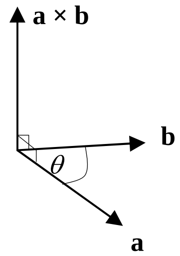
References:
Chapter 1
Pages 1-3

The vector product a × b

$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ where θ is the angle between **a** and **b** and $\hat{\mathbf{n}}$ is the unit vector perpendicular to **a** and **b** such that **a, b** and $\hat{\mathbf{n}}$ form a right-handed set of vectors.



Note that **a** × **a** = 0

E.g. Find the magnitude of $\mathbf{a}\times\mathbf{b}$ when

$$\mathbf{a} = \begin{pmatrix}1\\2\\1\end{pmatrix},\ \mathbf{b} = \begin{pmatrix}2\\0\\1\end{pmatrix},$$

$$\mathbf{a}\times\mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\hat{\mathbf{n}} = \sqrt{6}\sqrt{5}\sin\theta\hat{\mathbf{n}}$$

From the scalar product,

$$\mathbf{a}.\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = 2+0+1 = 3$$

$$\Rightarrow \cos\theta = \frac{3}{\sqrt{30}} \Rightarrow \sin\theta = \sqrt{1-\frac{9}{30}} = \sqrt{\frac{7}{10}}$$

$$\Rightarrow |\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta = \sqrt{30}\sqrt{\frac{7}{10}} = \sqrt{21}$$

References:
Chapter 1
Pages 4-7

Properties

The vector product is anti-commutative.

$\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$

(N.B. Commutative means that either order of a binary operation gives the same result. So addition is commutative, since 3 + 2 = 2 + 3, but subtraction is anti-commutative since 3 –2 = –(2 –3).)

Example 1.4
Page 7

If **a** and **b** are parallel then **a × b = 0.**
If either or both **a** and **b** are 0 then **a × b = 0.**
Note that **a × b = 0** does not mean that either **a** or **b** are 0 – they may be parallel.

$(m\mathbf{a})\times(n\mathbf{b}) = mn(\mathbf{a}\times\mathbf{b})$

$\mathbf{a}\times(\mathbf{b}+\mathbf{c}) = \mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$
This is the distributive law – the vector product is distributive over addition and subtraction.

E.g. If $\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} \neq 0$ prove that **a** + **c** is parallel to **b**.

$\mathbf{a}\times\mathbf{b} = \mathbf{b}\times\mathbf{c} \Rightarrow \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{c} = 0$
$\Rightarrow \mathbf{a}\times\mathbf{b} + \mathbf{c}\times\mathbf{b} = 0$
$\Rightarrow (\mathbf{a}+\mathbf{c})\times\mathbf{b} = 0$
i.e. **a** + **c** is parallel to **b**.

E.g. Prove that $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = 2\mathbf{a}\times\mathbf{b}$.

$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b}) = (\mathbf{a}-\mathbf{b})\times\mathbf{a} + (\mathbf{a}-\mathbf{b})\times\mathbf{b}$
$= \mathbf{a}\times\mathbf{a} - \mathbf{b}\times\mathbf{a} + \mathbf{a}\times\mathbf{b} - \mathbf{b}\times\mathbf{b}$
$= -\mathbf{b}\times\mathbf{a} + \mathbf{a}\times\mathbf{b}$ since $\mathbf{a}\times\mathbf{a} = \mathbf{b}\times\mathbf{b} = 0$
$= \mathbf{a}\times\mathbf{b} + \mathbf{a}\times\mathbf{b} = 2\mathbf{a}\times\mathbf{b}$

Base Vectors

The unit vectors parallel to the coordinate axes are **i**, **j** and **k**.

$\mathbf{i}\times\mathbf{j} = \mathbf{k},\ \mathbf{j}\times\mathbf{k} = \mathbf{i},\ \mathbf{k}\times\mathbf{i} = \mathbf{j}$
$\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = 0$

E.g. Calculate $\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\-1\\4\end{pmatrix}$.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\times\begin{pmatrix}2\\-1\\4\end{pmatrix} = \begin{pmatrix}2\times4-3\times-1\\3\times2-1\times4\\1\times-1-2\times2\end{pmatrix} = \begin{pmatrix}11\\2\\-5\end{pmatrix}$$

References:
Chapter 1
Pages 3-4

Example 1.2
Page 6

Exercise 1A
Q. 1(i), 5(i), 8

Component Form

$$\text{If } \mathbf{a} = \begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix},\ \mathbf{b} = \begin{pmatrix}b_1\\b_2\\b_3\end{pmatrix},\ \mathbf{a}\times\mathbf{b} = \begin{pmatrix}a_2b_3-a_3b_2\\a_3b_1-a_1b_3\\a_1b_2-a_2b_1\end{pmatrix}$$

E.g. Find the equation of the plane ABC, where the coordinates of A, B and C are (1,0,1), (2,1,1) and (3, 1, –1) respectively.

$$\mathbf{n} = \overrightarrow{AB}\times\overrightarrow{AC} \text{ where } \overrightarrow{AB} = \begin{pmatrix}1\\1\\0\end{pmatrix} \text{ and } \overrightarrow{AC} = \begin{pmatrix}2\\1\\-2\end{pmatrix}$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix}1\\1\\0\end{pmatrix}\times\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}-2\\2\\-1\end{pmatrix}$$

So equation of plane is $-2x+2y-z+d=0$
and is satisfied by A, giving $-2-1+d=0$
$\Rightarrow d = 3$
$\Rightarrow -2x+2y-z+3=0$
(You can check that B and C also satisfy this equation.)

References: Chapter 1 Pages 9-11

Example 1.5 Page 9

Exercise 1B Q. 1(i), (iii)

Intersection of two planes

The line of intersection of two planes lies in both planes and is therefore perpendicular to the perpendicular of both planes

If the normal direction of L_1 is $\mathbf{n_1}$ and the normal direction of L_2 is $\mathbf{n_2}$ then the direction of the intersecting line is $\mathbf{n_1} \times \mathbf{n_2}$.

It remains to find one point common to both planes, which can be done by putting, say, $z = 0$ and solving the equations simultaneously.

If $z = 0$ nowhere on the planes then the resulting equations will be inconsistent and so you should try, say, $x = 0$.

E.g. Find the line of intersection of the planes:

$L_1 : 3x + y - z = 4$

$L_2 : 2x - y + 3z = 6$

$$\mathbf{n_1} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 2 \\ -11 \\ -5 \end{pmatrix}$$

Substitute $z = 0 \Rightarrow 3x + y = 4,\ 2x - y = 6$

$\Rightarrow 5x = 10 \Rightarrow x = 2, y = -2$

$$\Rightarrow \frac{x-2}{2} = \frac{y+2}{-11} = \frac{z}{-5}$$

References: Chapter 1 Page 11

Example 1.6 Page 12

Exercise 1B Q. 2(i), (iii)

Angle between two planes

The angle between two vectors is found from the Scalar Product (covered in C4.)

The angle between two planes is the angle between their normal directions.

If the normal direction of L_1 is $\mathbf{n_1}$ and the normal direction of L_2 is $\mathbf{n_2}$ then the angle between the planes is the angle between these directions which is given by $\mathbf{n_1} . \mathbf{n_2} = |\mathbf{n_1}||\mathbf{n_2}|\cos\theta$

E.g. Find the angle between the planes defined in the example above.

$L_1 : 3x + y - z = 4$

$L_2 : 2x - y + 3z = 6$

$$\mathbf{n_1} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \Rightarrow \mathbf{n_1}.\mathbf{n_2} = 6 - 1 - 3 = 2$$

$|\mathbf{n_1}| = \sqrt{9+1+1} = \sqrt{11},\ |\mathbf{n_2}| = \sqrt{4+1+9} = \sqrt{14}$

$$\Rightarrow \cos\theta = \frac{2}{\sqrt{11}\sqrt{14}} \approx 0.1612 \Rightarrow 80.7^0$$

References: Chapter 1 Page 12

Exercise 1B Q. 4

Family (or sheaf) of planes

If $\pi_1 = 0$ is the equation of one plane and $\pi_2 = 0$ another such that the line of intersection is $l = 0$, Then the equation $\lambda\pi_1 + \mu\pi_2 = 0$ is the general equation of a family of planes with the common line l.

This is because any point on the line satisfies $\pi_1 = 0$ and $\pi_2 = 0$ and therefore $\lambda\pi_1 + \mu\pi_2 = 0$ for all values of λ and μ.

E.g. Find the equation of the plane that passes through the point (1, 2, 3) and the line found in the first example above.

$\pi_1 : 3x + y - z - 4 = 0$

$\pi_2 : 2x - y + 3z - 6 = 0$

Any plane through the line of intersection of these planes is given by $\lambda\pi_1 + \mu\pi_2 = 4\lambda + 6\mu$

$\Rightarrow (3\lambda + 2\mu)x + (\lambda - \mu)y + (-\lambda + 3\mu)z = 4\lambda + 6\mu$

This is satisfied by the point (1, 2, 3)

$\Rightarrow 3\lambda + 2\mu + 2\lambda - 2\mu - 3\lambda + 9\mu = 4\lambda + 6\mu \Rightarrow -2\lambda + 3\mu = 0$

A solution to this equation is $\mu = 2, \lambda = 3$

$\Rightarrow 13x + y + 3z = 24$

References: Chapter 1 Pages 14-16

Example 1.8 Page 15

Exercise 1C Q. 1(i), (ii), 3, 6

Intersection of two lines

If the lines are $l_1 = 0$ and $l_2 = 0$ then the line $l_1 = 0$ will be defined by one point, p, and a direction, $\mathbf{n_1}$. Any point on this line is then given by $p + \lambda\mathbf{n}_1$. Similarly, any point on $l_2 = 0$ is given by $q + \mu\mathbf{n}_2$.

Equating these two will give three equations in two unknowns.

Find the values of λ and μ from the first two and check for consistency in the third. If they are consistent then the values of λ and μ will give the point of intersection; if they are not consistent then the lines do not meet.

E.g. You are given $l_1 : \frac{x-1}{1} = \frac{y+1}{2} = \frac{z}{3}$

$l_2 : \frac{x+4}{3} = \frac{y-3}{-1} = \frac{z-1}{1}$ $l_3 : \frac{x-2}{2} = \frac{y-2}{1} = \frac{z+1}{-1}$

Show that l_1 and l_2 intersect but l_1 and l_3 do not.

Any point on l_1 is $(1+\lambda, -1+2\lambda, 3\lambda)$

Any point on l_2 is $(-4+3\mu, 3-\mu, 1+\mu)$

These two points are the same if $1+\lambda = -4+3\mu$ and $-1+2\lambda = 3-\mu$

$\Rightarrow \lambda - 3\mu = -5$ and $2\lambda + \mu = 4 \Rightarrow \lambda = 1, \mu = 2$

$\Rightarrow$ The x and y values are the same and substituting into the z coordinates also gives the same value.

So the lines meet at (2,1,3).

Any point on l_3 is $(2+3\nu, 2+\nu, -1+\nu)$

Solving for equal x and y values gives $\lambda = \frac{8}{5}, \nu = \frac{1}{5}$

but the z values are not the same so the lines do not intersect.

References:
Chapter 1
Pages 19-22

Example 1.10
Page 22

Exercise 1D
Q. 1(i), 2(ii)

Distance of a point from a line.
If P is a point not on a line l and A is any point on the line and M is the closest point on the line from P, then the distance is the length of the line PM.

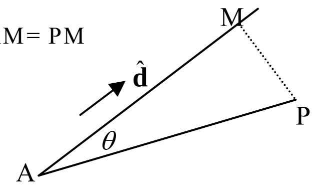
If the direction of l is defined by the unit vector, $\hat{\mathbf{d}}$, then PM = APsinPAM.

Since $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta$, take $\mathbf{b} = \mathbf{AP}$ and $\mathbf{a} = \hat{\mathbf{d}}$.

Then $\left|\hat{\mathbf{d}}\times\overrightarrow{AP}\right| = |\hat{\mathbf{d}}|\left|\overrightarrow{AP}\right|\sin PAM$

$= \left|\overrightarrow{AP}\right|\sin PAM = PM$

So $\left|\overrightarrow{PM}\right| = \left|\hat{\mathbf{d}}\times\overrightarrow{AP}\right|$



E.g. Find the distance of the point (1, 2, 3) from the line $\frac{x-2}{2} = \frac{y+3}{-1} = \frac{z+1}{3}$

$$\mathbf{d} = \begin{pmatrix}2\\-1\\3\end{pmatrix} \Rightarrow |\mathbf{d}| = \sqrt{14} \Rightarrow \hat{\mathbf{d}} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\-1\\3\end{pmatrix}$$

Take A (2, −3, −1) and P (1, 2, 3) $\Rightarrow \overrightarrow{AP} = \begin{pmatrix}-1\\5\\4\end{pmatrix}$

$$\hat{\mathbf{d}}\times\overrightarrow{AP} = \frac{1}{\sqrt{14}}\begin{pmatrix}2\\-1\\3\end{pmatrix}\times\begin{pmatrix}-1\\5\\4\end{pmatrix} = \frac{1}{\sqrt{14}}\begin{pmatrix}-19\\-11\\9\end{pmatrix}$$

$$\Rightarrow \left|\hat{\mathbf{d}}\times\overrightarrow{AP}\right| = \frac{1}{\sqrt{14}}\sqrt{19^2+11^2+9^2} = \sqrt{\frac{563}{14}}$$

References:
Chapter 1
Page 26-27

Exercise 1D
Q. 3(i), 4(ii)

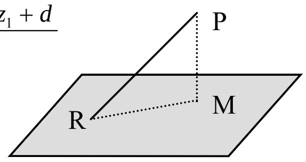
Distance of a point from a plane.
If M is the foot of the perpendicular from P(x_1, y_1, z_1) to a plane then the distance of P from the plane is PM.
The direction of PM is the normal direction of the plane, **n**.
Let the equation of the plane be $ax + by + cz + d = 0$
Take any point, R, on the plane. In general this can be (x_2, y_2, z_2), but if c is not zero then this can be (0, 0, $-{}^{d}/_{c}$).
Let the angle between PR and PM be θ.

Then the scalar product gives $\overrightarrow{PR}.\hat{\mathbf{n}} = |PR|\cos\theta$.

and PM = PR$\cos\theta \Rightarrow |PM| = \left|\overrightarrow{PR}.\hat{\mathbf{n}}\right|$

$$\overrightarrow{RP} = \begin{pmatrix}x_1-x_2\\y_1-y_2\\z_1-z_2\end{pmatrix},\quad \mathbf{n} = \begin{pmatrix}a\\b\\c\end{pmatrix}$$

$$\Rightarrow \overrightarrow{RP}.\mathbf{n} = a(x_1-x_2)+b(y_1-y_2)+c(z_1-z_2)$$
$$= ax_1+by_1+cz_1+d \text{ since } ax_2+by_2+cz_2+d = 0$$
$$\Rightarrow \overrightarrow{RP}.\hat{\mathbf{n}} = \frac{ax_1+by_1+cz_1+d}{|\mathbf{n}|}$$
$$\Rightarrow \text{distance} = \left|\overrightarrow{PR}.\hat{\mathbf{n}}\right|$$
$$= \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$



Note that:
If two distances are opposite signs then the points are on opposite sides of the plane.
If the distance is 0 then the point lies on the plane.

E.g. Find the distance of the point (1, 2, 1) from the plane $2x + 3y - z = 4$

$$\text{Distance} = \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right| = \frac{2\times1+3\times2-1-4}{\sqrt{2^2+3^2+1^2}}$$
$$= \frac{3}{\sqrt{14}}$$

E.g. Show that the points (6, 2, 2) and (2, -4, 4) are equidistant from the plane $2x + 3y - z - 2 = 0$

$$\text{Distance} = \left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$$

For (6, 2, 2), $d = \frac{2\times6+3\times2-1\times2-2}{\sqrt{2^2+3^2+1^2}} = \frac{14}{\sqrt{14}}$

For (2, −4, 4), $d = \frac{2\times2-3\times4-1\times4-2}{\sqrt{2^2+3^2+1^2}} = \frac{-14}{\sqrt{14}}$

So same distance but opposite sides.

E.g. Find the foot of the perpendicular from the point P(3, 5, 4) to the line $\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\1\end{pmatrix}$.

Any point, A, on the line is $(1+2\lambda, 2-\lambda, 3+\lambda)$.

$$\overrightarrow{PA} = \begin{pmatrix}1+2\lambda-3\\2-\lambda-5\\3+\lambda-4\end{pmatrix} = \begin{pmatrix}2\lambda-2\\-\lambda-3\\\lambda-1\end{pmatrix}.$$

This direction is perpendicular to the line.

$$\Rightarrow \begin{pmatrix}2\lambda-2\\-\lambda-3\\\lambda-1\end{pmatrix}.\begin{pmatrix}2\\-1\\1\end{pmatrix} = 0$$

$$\Rightarrow 4\lambda-4+\lambda+3+\lambda-1 = 0 \Rightarrow \lambda = \frac{1}{3}$$

$$\Rightarrow \text{A is } \left(1\frac{2}{3}, 1\frac{2}{3}, 3\frac{1}{3}\right)$$

References:
Chapter 1
Pages 31-36

The Scalar Triple Product

Given that $\mathbf{b} \times \mathbf{c} = |\mathbf{b}||\mathbf{c}|\sin\theta\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{b}$ and $\mathbf{c}$,

and $\mathbf{a}.\hat{\mathbf{n}} = |\mathbf{a}||\hat{\mathbf{n}}|\cos\varphi$ with $|\hat{\mathbf{n}}| = 1$

$$\mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = (\mathbf{a}\,.\hat{\mathbf{n}})\,|\mathbf{b}||\mathbf{c}|\sin\theta = |\mathbf{a}||\mathbf{b}||\mathbf{c}|\sin\theta\cos\varphi$$
$$= (\mathbf{a}\,.\hat{\mathbf{n}})\,|\mathbf{b} \times \mathbf{c}\,|$$

The Scalar Triple Product in component form.

If $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$,

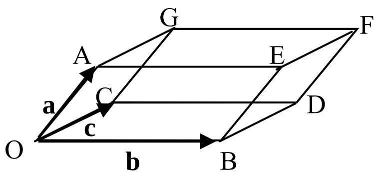
$$\mathbf{b} \times \mathbf{c} = \begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}, \Rightarrow \mathbf{a}\,.\,\mathbf{b} \times \mathbf{c} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}.\begin{pmatrix} b_2c_3 - b_3c_2 \\ b_3c_1 - b_1c_3 \\ b_1c_2 - b_2c_1 \end{pmatrix}$$
$$= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$
$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Exercise 1E
Q. 1(i), 2(i)

Volume of the parallelepiped OAEBCGFD

where the three sets of parallel sides are given by
OA = **a**, OB = **b**, OC = **c**
is given by

$$V = |\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$$



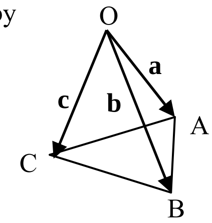
Example 1.12
Page 33

If the volume is 0 then the four points lie on the same plane.

Volume of the tetrahedron OABC

where the three sides are given by
OA = **a**, OB = **b**, OC = **c**

is given by $V = \frac{1}{6}|\mathbf{a}.(\mathbf{b} \times \mathbf{c})|$



References:
Chapter 1
Pages 24-25

Distance between two skew lines

If l_1 is defined by a point **a** and direction $\mathbf{n_1}$ and l_2 by a point **b** and direction $\mathbf{n_2}$ then the shortest distance is given by

$$d = \left| \frac{(\mathbf{a} - \mathbf{b}).(\mathbf{n}_1 \times \mathbf{n}_2)}{|\mathbf{n}_1 \times \mathbf{n}_2|} \right|$$

If the shortest distance is 0 then the lines intersect.

Exercise 1F
Q. 2, 3

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Competence statements v8, v9, v10, v11, v12

E.g. Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$, $\mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$,

find $\mathbf{a}.\mathbf{b}\times\mathbf{c}$.

$$\mathbf{a}.\mathbf{b}\times\mathbf{c} = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ -1 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} + 1\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$
$$= 1\times(-4) + 3(-5) + 1(3) = -16$$

Show that the points A(1, 1, 1), B(2, 4, 7), C(−1, 3, 1) and D(3, 1, 4) are coplanar.

$$\text{AB} = \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix}, \text{AC} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}, \text{AD} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$$

$$\text{AB} \times \text{AC} = \begin{pmatrix} -12 \\ -12 \\ 8 \end{pmatrix}$$

$$\text{AD.AB} \times \text{AC} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} . \begin{pmatrix} -12 \\ -12 \\ 8 \end{pmatrix} = -24 + 0 + 24 = 0$$

E.g. You are given the equations of two lines:

$$l_1 : \frac{x-1}{3} = \frac{y-2}{1} = \frac{z-1}{2}$$
$$l_2 : \frac{x-3}{2} = \frac{y-1}{-1} = \frac{z-k}{1}$$

(i) Find the distance between the lines when $k = 9$.

(ii) Find the value of k if the two lines intersect.

$$\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ 1 \\ k \end{pmatrix} \Rightarrow \mathbf{a} - \mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ 1-k \end{pmatrix}$$

$$\mathbf{n_1} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \mathbf{n_1} \times \mathbf{n_2} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix},$$

$$|\mathbf{n_1} \times \mathbf{n_2}| = \sqrt{9+1+25} = \sqrt{35}$$

$$(\mathbf{a} - \mathbf{b}).(\mathbf{n_1} \times \mathbf{n_2}) = \begin{pmatrix} -2 \\ 1 \\ 1-k \end{pmatrix} \times \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} = 5k - 10$$

$$\Rightarrow d = \frac{5k-10}{\sqrt{35}}$$

(i) When $k = 9, d = \sqrt{35}$

(ii) When $d = 0, 5k - 10 = 0 \Rightarrow k = 2$

References:
Chapter 2
Pages 43-50

Example 2.1
Page 45

Exercise 2A
Q. 2, 7

Example 2.2
Page 49

Exercise 2B
Q. 2, 3

A function of three variables

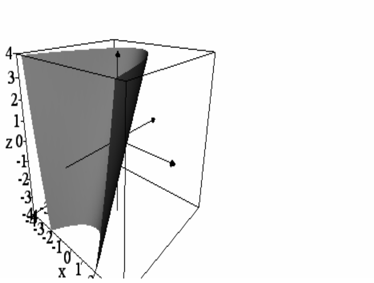
Just as the function $y = \mathrm{f}(x)$ represents a curve in two dimensions, the function $z = \mathrm{f}(x, y)$ represents a surface.

If the x- and y-axes are horizontal and the z-axis is vertical then for any coordinate pair (x, y) a value of z can be found.

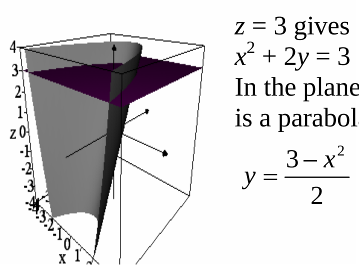
All the points where z is equal is known as a **contour.**

A vertical plane cuts the surface in what is called a **section.**

E.g. $z = x^2 + 2y$

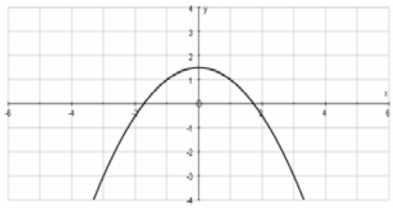


E.g. Determine the section $z = 3$ of the above function.



$z = 3$ gives
$x^2 + 2y = 3$
In the plane this is a parabola

$$y = \frac{3 - x^2}{2}$$



References:
Chapter 2
Pages 52-53

Example 2.4
Page 53

Exercise 2C
Q. 1(i), (iii), 5

Partial differentiation

This is the process of differentiating the function $z = \mathrm{f}(x, y)$ with respect to x keeping y constant and differentiating $z = \mathrm{f}(x, y)$ with respect to y, keeping x constant.

E.g. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = x^2 + 3x - y$

$\frac{\partial z}{\partial x} = 2x + 3, \ \frac{\partial z}{\partial y} = -1$

References:
Chapter 2
Pages 55-59

Example 2.5
Page 58

Differentiability- The Tangent Plane

At a point on a continuous surface the plane which touches the surface at a point is said to be the tangent plane at that point.
It contains the tangent of the section of the surface parallel to the x-axis at that point and also the tangent of the section of the surface parallel to the y-axis at that point.

The directions of these lines are given by $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$\frac{\partial z}{\partial x} = c$ means that from the point, the change in y is zero (because we keep it constant!)
and for a change of 1 in x there is a change of c in z.

E.g. Find the equation of the tangent plane to the surface $z = x^2 + 2xy + 4x$ at the point (1, 2, 9).

$$\frac{\partial z}{\partial x} = 2x + 2y + 4 = 10 \text{ giving the direction } \begin{pmatrix} 1 \\ 0 \\ 10 \end{pmatrix}$$

$$\frac{\partial z}{\partial y} = 2x = 2 \text{ giving the direction } \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow \mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 10 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -10 \\ -2 \\ 1 \end{pmatrix}$$

So the plane can be written $10x + 2y - z = d$
and the equation is satisfied by the point (1, 2, 9)
$\Rightarrow d = 10 + 4 - 9 = 5$
$\Rightarrow 10x + 2y - z = 5$

Exercise 2D
Q. 1(i), (ii)

Alternative notation

If $z = \mathrm{f}(x, y)$ then differentiating with respect to x gives $\frac{\partial z}{\partial x}$.

This can also be written $\frac{\partial \mathrm{f}}{\partial x}$ or $\mathrm{f}_x(x, y)$.

References: Chapter 2 Pages 61-63

Exercise 2E Q. 1, 3

Directional derivatives

In any 3-D representation, the horizontal direction of a line may be denoted by the unit vector $\hat{\mathbf{u}} = \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$, where the line makes an angle of α with the x-axis.

Then $\hat{\mathbf{u}} \bullet \begin{pmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{pmatrix} = \hat{\mathbf{u}} \bullet \mathbf{grad}\ \mathrm{f} = \frac{\partial z}{\partial x}\cos\alpha + \frac{\partial z}{\partial y}\sin\alpha$ is the directional derivative.

E.g. Find grad f when f = $2x + xy^2$.
Find the gradient on this surface at (1, 1, 3) in the direction $\hat{\mathbf{u}} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$.

$\frac{\partial \mathrm{f}}{\partial x} = 2 + y^2, \quad \frac{\partial \mathrm{f}}{\partial y} = 2xy$

$\mathbf{grad}\ \mathrm{f} = \begin{pmatrix} 2+y^2 \\ 2xy \end{pmatrix}$. At A, $\mathbf{grad}\ \mathrm{f} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

Gradient = $\begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix} . \begin{pmatrix} 3 \\ 2 \end{pmatrix} = 1.8 + 1.6 = 3.4$

References: Chapter 2 Pages 64-66

Example 2.7 Page 65

Exercise 2F Q. 2, 3

Stationary points

A stationary point on a surface is defined as a point where the tangent plane is parallel to the x-y plane. This is a point which is a local maximum or minimum of z.

This occurs when $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$.

The nature of a stationary point may be determined by considering sections or the value of z for small changes in x and y.

E.g. Investigate the stationary points on the curve $z = x^2 - 8xy + 2y^2 + 14x$.

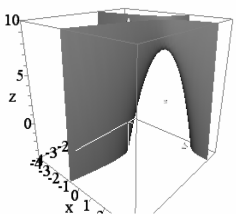
$z = x^2 - 8xy + 2y^2 + 14x$

$\Rightarrow \frac{\partial z}{\partial x} = 2x - 8y + 14, \quad \frac{\partial z}{\partial y} = -8x + 4y$

$2x - 8y = -14$

$-8x + 4y = 0$

$\Rightarrow x = 1, y = 2, z = 7$



References: Chapter 3 Pages 69-72

Example 2.9 Page 71

Exercise 2G Q. 3, 4

Small changes

For $z = \mathrm{f}(x, y)$, $\delta z \approx \frac{\partial \mathrm{f}}{\partial x}\delta x + \frac{\partial \mathrm{f}}{\partial y}\delta y$

This formula is applicable for any number of variables.

For $z = \mathrm{f}(x, y, w)$, $\delta z \approx \frac{\partial \mathrm{f}}{\partial x}\delta x + \frac{\partial \mathrm{f}}{\partial y}\delta y + \frac{\partial \mathrm{f}}{\partial w}\delta w$

The approximation can be used to estimate the effects of errors in a calculation.

E.g. in the example above, the stationary point is (1, 2, 7).

$z = x^2 - 8xy + 2y^2 + 14x$

When $y = 2$, $z = x^2 - 2x + 8$

$\Rightarrow \frac{\partial z}{\partial x} = 2x - 2 = 0$ at $x = 1$

Also, when $x = 1$, $z = 2y^2 - 8y + 15$

$\Rightarrow \frac{\partial z}{\partial y} = 4y - 8 = 0$ at $y = 2$

At this point $\frac{\partial^2 z}{\partial x^2} > 0$ and $\frac{\partial^2 z}{\partial y^2} > 0$

i.e. the stationary value is a minimum.

References: Chapter 3 Pages 74-76

Example 2.11 Page 75

Exercise 2H Q. 1, 5

The Directional Derivative

If $w = \mathrm{g}(x, y, z)$ then the directional derivative is

$\hat{\mathbf{u}}.\mathbf{grad}$ g, where $\mathbf{grad}\ \mathrm{g} = \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial z} \end{pmatrix}$

References: Chapter 3 Pages 77-79

Example 2.12 Page 78

Exercise 2I Q. 1, 4

The surface g(*x*,*y*,*z*) = *k*.

For the point A with position vector **a** on the surface g(***x***,***y***,***z***) = k , the tangent plane is **(r –a).grad** g = 0, where **grad** g is evaluated at point A.

The normal line is **r** = **a** + λgrad **g.**

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Competence statements c4, c5, c6, c8

References: Chapter 3 Pages 85-91

Example 3.3 Page 89

Exercise 3A Q. 1, 2, 3

Envelopes

The **family** of lines obeying a rule is the set of equations f(x,y,p) = 0.

The equation of the **envelope** is given by the two equations

$$f(x, y, p) = 0, \frac{\partial}{\partial p} f(x, y, p) = 0$$

If p can be eliminated then the Cartesian equation results. Alternatively rearrange to give parametric equations x = g(p), y = h(p).

References: Chapter 3 Page 93-97

Example 3.4 Page 96

Exercise 3B Q. 1, 2, 6

Arc length

Cartesian coordinates: $y = f(x)$

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow s = \int_{x=a}^{x=b} \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) dx$$

Polar coordinates: $r = f(\theta)$

$$\frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} \Rightarrow s = \int_{\theta=a}^{\theta=b} \left(\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\right) d\theta$$

Parametric coordinates: $x = f(\theta),\ y = f(\theta)$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$\Rightarrow s = \int_{\theta=a}^{\theta=b} \left(\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\right) d\theta$$

References: Chapter 3 Page 99

Volume of solid of revolution

The volume of the solid swept out when the curve $y = f(x)$ is rotated through 2π about the x-axis is given by $V = \int_{x=a}^{x=b} \pi y^2 dx$.

The volume of the solid swept out when the curve $x = f(y)$ is rotated through 2π about the y-axis is given by $V = \int_{y=a}^{y=b} \pi x^2 dy$.

References: Chapter 3 Pages 100-102

Example 3.6 Page 101

Exercise 3C Q. 1(i), 5

Surface area of solid of revolution

The surface area of the solid swept out when the curve $y = f(x)$ is rotated through 2π about the x-axis is given by $S = \int_{x=a}^{x=b} 2\pi y\, ds$.

In cartesian coordinates this becomes

$$S = \int_{x=a}^{x=b} 2\pi y \frac{ds}{dx} dx$$

The surface area of the solid swept out when the curve $x = f(y)$ is rotated through 2π about the y-axis is given by $S = \int_{y=a}^{y=b} 2\pi x\, ds$.

In caresian coordinates this becomes

$$S = \int_{y=a}^{y=b} 2\pi x \frac{ds}{dy} dy$$

E.g. Find the circumference of a circle.

1. Cartesian coordinates:

$$x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

Length of positive quadrant ($x = 0$ to $x = a$)

$$s = \int_{x=0}^{x=a} \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) dx = \int_{x=0}^{x=a} \left(\sqrt{1+\left(\frac{x}{y}\right)^2}\right) dx$$

$$= \int_{x=0}^{x=a} \left(\sqrt{\frac{y^2+x^2}{y^2}}\right) dx = \int_{x=0}^{x=a} \left(\sqrt{\frac{a^2}{a^2-x^2}}\right) dx = a\int_{x=0}^{x=a} \frac{1}{\sqrt{a^2-x^2}} dx$$

$$\text{Let } x = a\sin\theta : \frac{dx}{d\theta} = a\cos\theta,\ a^2 - x^2 = a^2\cos^2\theta$$

$$\text{When } x = 0, \theta = 0;\ \text{when } x = a, \theta = \frac{\pi}{2}$$

$$\Rightarrow s = a\int_{\theta=0}^{\theta=\pi/2} d\theta = \frac{a\pi}{2}$$

So for whole circle, $c = 4 \times \frac{a\pi}{2} = 2a\pi$

2. Parametric coordinates:

$x = a\cos\theta,\ y = a\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \frac{dy}{d\theta} = a\cos\theta; \frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}$$

$$\Rightarrow \frac{ds}{d\theta} = \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a$$

$$\Rightarrow s = [a\theta]_0^{2\pi} = 2a\pi$$

3. Polar coordinates

$$r = a \Rightarrow \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = a \Rightarrow s = \int_{\theta=0}^{\theta=2\pi} a\, d\theta$$

$$= [a\theta]_0^{2\pi} = 2a\pi$$

E.g. To find the surface area of a sphere.

Rotate a circle through 360^0 about the x-axis.

$x^2 + y^2 = a^2$ between $x = -a$ and $x = a$

$$\Rightarrow S = \int_{-a}^{a} 2\pi y \frac{ds}{dx} dx \text{ where } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$x^2 + y^2 = a^2 \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{x}{y}\right)^2 = \frac{x^2+y^2}{y^2} = \frac{a^2}{y^2}$$

$$\Rightarrow \frac{ds}{dx} = \frac{a}{y} \Rightarrow S = \int_{-a}^{a} 2\pi y.\frac{a}{y} dx = \int_{-a}^{a} 2\pi a\, dx = 2\pi a[x]_{-a}^{a}$$

$$\Rightarrow S = 4\pi a^2$$

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Summary FP3 Option 3: **Differentiable Geometry - 2**

References:
Chapter 3
Pages 105-108

Example 3.9
Page 108

Exercise 3D
Q. 1, 3

Intrinsic Equations

An alternative way to describe a curve is in terms of the arc length, *s*, with the angle ψ, which its tangent makes with a fixed direction. We determine the equation uniquely we also need the point of the curve where *s* = 0, the direction where $\psi = 0$ and also a sense of direction. (ψ is usually measured in radians anticlockwise.)

$$\tan\psi = \frac{\mathrm{d}y}{\mathrm{d}x}, \quad \frac{\mathrm{d}x}{\mathrm{d}s} = \cos\psi, \quad \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\psi$$

References:
Chapter 3
Page 109-112

Example 3.11
Page 111

Exercise 3E
Q. 1, 6

Curvature

The curvature of a curve at a point P is the rate of change of ψ with *s* at P.

$$\kappa = \frac{\mathrm{d}\psi}{\mathrm{d}s}$$

If κ is positive , then ψ increases with *s* and the curve curves to the left.

For a curve given in intrinsic form the formula above can be used.
If the equation is given in Cartesian coordinates, *y* = f(*x*) then

$$\kappa = \frac{\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}}{\left(1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}$$

References:
Chapter 3
Page 115-117

Example 3.12
Page 116

Exercise 3F
Q. 1, 2

Centre of curvature

The circle of curvature of a curve at a point P is the circle with centre on the normal at P.

The radius is $\rho = \frac{1}{\kappa} = \frac{\mathrm{d}s}{\mathrm{d}\psi}$

We define unit vectors in the direction of the positive tangent and positive normal to be $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ where $\hat{\mathbf{t}} = \begin{pmatrix}\cos\psi\\ \sin\psi\end{pmatrix}$ and $\hat{\mathbf{n}} = \begin{pmatrix}-\sin\psi\\ \cos\psi\end{pmatrix}$

Then, if the vector for P is $\mathbf{r}$ and the vector for the centre, C, is $\mathbf{c}$ then $\mathbf{c} = \mathbf{r} + \rho\hat{\mathbf{n}}$

References:
Chapter 3
Page 118-119

Example 3.13
Page 118

Exercise 3G
Q. 1, 4

The Evolute of a Curve

As the point P moves along a curve , the centre of curvature, C, also moves. The locus of C is called the *evolute* of the curve.
As shown, the centre of curvature can be found for a specific point. If, instead, the parametric point is used then the form of **c** will be in parametric form, which will be the equation of the evolute.

An alternative form is $\frac{\mathrm{d}\mathbf{c}}{\mathrm{d}s} = \frac{\mathrm{d}\rho}{\mathrm{d}s}\hat{\mathbf{n}}$.

E.g. The curve with intrinsic equation $s = 4a(1-\cos\psi)$ has a stationary point at the origin.
Find the paramtric equations for the curve.

$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\psi, \ \frac{\mathrm{d}y}{\mathrm{d}s} = \sin\psi$. From the curve $\frac{\mathrm{d}s}{\mathrm{d}\psi} = 4a\sin\psi$

$$\frac{\mathrm{d}x}{\mathrm{d}\psi} = \frac{\mathrm{d}x}{\mathrm{d}s}\times\frac{\mathrm{d}s}{\mathrm{d}\psi} = \cos\psi\times 4a\sin\psi = 2a\sin 2\psi$$

$\Rightarrow x = k - a\cos 2\psi;\ \ x = 0$ when $\psi = 0 \Rightarrow k = a$

$\Rightarrow x = a(1-\cos 2\psi)$

$$\frac{\mathrm{d}y}{\mathrm{d}\psi} = \frac{\mathrm{d}y}{\mathrm{d}s}\times\frac{\mathrm{d}s}{\mathrm{d}\psi} = \sin\psi\times 4a\sin\psi = 2a(1-\cos 2\psi)$$

$\Rightarrow y = k + 2a\psi - a\sin 2\psi;\ y = 0$ when $\psi = 0 \Rightarrow k = 0$

$\Rightarrow y = a(2\psi - \sin 2\psi)$

Writing $2\psi = \theta$ gives parametric equations

$x = a(1-\cos\theta),\ y = a(\theta - \sin\theta)$

E.g. For the point P $\left(a, \frac{1}{4}a\right)$ on the curve $4a^2 y = x^3$ (where a is a positive constant), find
(i) the radius of curvature,
(ii) the coordinates of the centre of curvature.

(i) At P, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x^2}{4a^2} = \frac{3}{4}, \ \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = \frac{6x}{4a^2} = \frac{3}{2a}$

$$\Rightarrow \rho = \frac{\left(1+\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{3/2}}{\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}} = \frac{\left(1+\left(\dfrac{3}{4}\right)^2\right)^{3/2}}{\dfrac{3}{2a}}$$

$$= \frac{\left(\dfrac{5}{4}\right)^3}{\dfrac{3}{2a}} = \frac{125}{64}\times\frac{2a}{3} = \frac{125a}{96}$$

(ii) Normal vector is $\begin{pmatrix}-3\\ 4\end{pmatrix} \Rightarrow \hat{n} = \frac{1}{5}\begin{pmatrix}-3\\ 4\end{pmatrix}$

$$\Rightarrow \text{ Centre of curvature is } \begin{pmatrix}a\\ \frac{1}{4}a\end{pmatrix} + \frac{125a}{96}\times\frac{1}{5}\begin{pmatrix}-3\\ 4\end{pmatrix}$$

$$\text{i.e. } \begin{pmatrix}a - \dfrac{75a}{96}\\ \dfrac{1}{4}a + \dfrac{100a}{96}\end{pmatrix} \quad \text{which is } \left(\frac{7a}{32}, \frac{31a}{24}\right)$$

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Competence statements g4, g5, g6, g7, g8

References: Chapter 4 Pages 130-132

Sets and operations
A **set** is a collection of items having a common property.
A **binary operation** is an operation combining two items of a set to form a third item.
The result of a binary operation is often referred to as the **product** (though most people restrict this word to the result of the binary operation "multiply".)
The operation is **closed** with respect to a set if, for all elements x, y of the set, the product $x*y$ lies in the set.
The operation is **commutative** if, for all $x, y \in S$, $x*y = y*x$.
The operation is **associative** if, for all $x, y, z \in S$ $x*(y*z) = (x*y)*z$
An **identity element**, $e \in S$, is an element such that $e*x = x * e = x$ for all $x \in S$.
The **inverse** x^{-1} , of an element x is an element such that $x * x^{-1} = x^{-1} * x = e$

References: Chapter 4 Page 132

Exercise 4A Q. 2, 3

Modular arithmetic
Within the set of integers, two numbers are said to be **congruent modulo *m*** if the difference between them is a multiple of m.
E.g. $37 = 1 \pmod 3$ because $37 - 1 = 3 \times 12$.
In modulo 3 arithmetic all integers can be reduced to the numbers 0, 1 or 2.

References: Chapter 4 Page 135-143

Exercise 4B Q. 1, 2

Groups
A **Group** $(S,*)$ is a non-empty set S with a binary operation * such that
* is closed in S - i.e. for all $x, y \in S$, $x*y \in S$
* is associative in S i.e. for all $x, y, z \in S$, $x*(y*z) = (x*y)*z$

There is an **identity element**, $e \in S$ such that $e*x = x * e = x$ for all $x \in S$
For every element of the set, x, there exists an **inverse element** $x^{-1} \in S$ such that $x * x^{-1} = x^{-1} * x = e$
An element that is its own inverse is said to be **self-inverse.**

If, in addition, the operation is commutative, then the group is said to be **Abelian**.

Example 4.2 Page 140

The table showing the combination of elements is called the **Cayley Table.** In each row and column each element will occur once and once only.

The order of a Group
The order of a finite group is the number of elements in the group.

The order of an element, x, is the smallest positive integer n such that $x^n = e$.

Exercise 4C Q. 1, 2, 3

Properties of a Group
The identity element is unique.

Each element has an unique inverse.

If $x*y = x*z$ then $y = z$ (known as the cancellation law)
The equation $ax = b$ has the unique solution $x = a^{-1}b$.

E.g. The binary operation "add" is closed with respect to the set of positive numbers because the addition of any two positive numbers is positive. The binary operation "subtract" however is not closed. For example $4 - 5$ is not a positive number.

E.g. The binary operation "add" is commutative because the addition of any two positive numbers is same whichever way round you combine the numbers. I.e. $4 + 5 = 5 + 4$. The binary operation "subtract" however is not commutative. For example $4 - 5 \neq 5 - 4$.

E.g. The binary operation "add" is associative:
E.g. $6 + (5 + 4) = 6 + 9 = 15$
and $(6 + 5) + 4 = 11 + 4 = 15$
The binary operation "subtract" however is not associative:
E.g. $6 - (5 - 4) = 6 - 1 = 5$
and $(6 - 5) - 4 = 1 - 4 = -3$

E.g. Consider the set G and the binary operation of multiplication modulo 20, where
$G = \{1, 3, 7, 9, 11, 13, 17, 19\}$
Show that G is a group under this operation.

The combination table is

	1	**3**	**7**	**9**	**11**	**13**	**17**	**19**
1	1	3	7	9	11	13	17	19
3	3	9	1	7	13	19	11	17
7	7	1	9	3	17	11	19	13
9	9	7	3	1	19	17	13	11
11	11	13	17	19	1	3	7	9
13	13	19	11	17	3	9	1	7
17	17	11	19	13	7	1	9	3
19	19	17	13	11	9	7	3	1

(i) The set is closed under the operation
(ii) Multiplication is associative
(iii) The identity element is 1
(iv) There is an inverse for each element (i.e. 1 appears in each row and each column).

E.g. State the order of G and find the order of each element.
The order of the group is the number of elements i.e. 8.
The order of each element, x, is the smallest integer, n, such that $x^n = 1$
For 1 the order is 1.
For 9, 11, 19, the order is 2 (These are the elements where 1 is in the leading diagonal)
For 3, 7, 13, 17, the order is 4.
(i.e. $3\times3 = 9$, $9\times3 = 7$, $7\times3 = 1$)

References:
Chapter 4
Page 146-149

Exercise 4D
Q. 1, 8

Isomorphism
Consider two groups with the same order.
If the mapping of the elements of one group to the other preserves the structure then the two groups are said to be **isomorphic**.

References:
Chapter 4
Pages 151-153

Example 4.4
Page 151

Exercise 4E
Q. 1, 6

Subgroups
A **subgroup** of a group (S, *) is a non-empty subset of S which forms a group under the operation *.

Every group has a trivial sub-group $\{e\}$. As with factors of a number, you may also consider the set as a subgroup of itself.

A **proper subgroup** is a subgroup that is not one of the above.

References:
Chapter 4
Pages 154-156

Exercise 4F
Q. 3, 4

Lagrange's Theorem
The order of any sub-group is a factor of the order of the group.

For instance, a group of order 4 can have subgroups only of order 1, 2 or 4, but not 3.

References:
Chapter 4
Pages 159-160

Example 4.5
Page 160

Exercise 4G
Q. 2, 5

Cyclic groups
If a member of a group is x then x^2, x^3, etc are also members of the group.

There must be a smallest number, m, such that $x^m = e$.
m must be less than or equal to n, the order of the group.

If $m = n$, then each member of the group is a power of x. x is said to **generate** the group and the group is said to be **cyclic.**

The group $\{e, a, a^2, a^3\}$ with $a^4 = e$ is cyclic.

Groups with order a prime number must be cyclic. This is because the order of each element is a factor of p, the order of the group (Lagrange's Theorem). Since e is the only element with order 1, all others much have order p and so must generate the group.

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Competence statements a3, a5, a6, a7, a8

E.g. List all the sub-groups of G in example on previous page.

The identity element {1} always forms a sub-group or order 1.

Other **proper subgroups** are found by scrutinising the combination table.
Any element with order 2 will, with e form a proper subgroup of order 2 if e is in the leading diagonal position for the element.
So {1,9}, {1,11} {1,19}are proper subgroups of order 2.

By Lagrange's Theorem there cannot be any subgroups of order 3.

Scrutiny of the combination table will reveal that the "top left" block of 4 elements contains only those 4 elements.
Therefore {1,3,7,9}is proper subgroups of order 4.
There are two others:
{1, 9, 13, 17}, {1, 9, 11, 19}
Note the combination table for these subgroups which are an extraction of the combination table of G.

	1	**9**	**13**	**17**
1	1	9	13	17
9	9	1	17	13
13	13	17	9	1
17	17	13	1	9

	1	**9**	**11**	**19**
1	1	9	11	19
9	9	1	19	11
11	11	19	1	9
19	19	11	9	1

The set G itself is a subgroup of itself, of order 8.

E.g. List the subgroups of G that are isomorphic to one another.

Subgroups of order 2 will always be isomorphic to one another.
I.e. {1, 9}, {1, 11} and {1, 19}

Likewise, subgroups of order 4 will be isomorphic to each other providing the identity element is in the same place within their tables.
This is so for {1, 3, 7, 9} and {1, 9, 13, 17}.
[The subgroup {1, 9, 11, 19} has the identity element in every place of the leading diagonal.]

E.g. Prove that G is not cyclic.

For all elements in G, there is a least value of n for which $x^n = 1$. We have seen above that the values of n for the elements are 1, 2 and 4.

In order for G to be cyclic there must be at least one element, x, for which $x^8 = 1$ with 8 the smallest such value.

References: Chapter 5 Pages 171-175

Terminology

A sequence of events where the probability of an outcome at one stage depends only on the outcome at the previous stage is known as a **Markov Chain**.
The conditional probabilities of passing from one stage to the next are called **transition probabilities**.
They are most usefully arranged in a square **transition matrix, P.** Each column of **P** is a **probability vector.** It follows that the sum of elements of each column is 1.
If the column vector **p** represents the probabilities at one stage and **P** is the transition matrix then **Pp** represents the probabilities at the next stage.
E.g. if there are, at any stage, two outcomes then **P** is a 2 × 2 matrix.
If the two outcomes are A and B, then **P** is given by

$$\mathbf{P} = \begin{pmatrix} P(A|A) & P(A|B) \\ P(B|A) & P(B|B) \end{pmatrix}$$

Exercise 5A Q. 1(i), 5

E.g. $\mathbf{P} = \begin{pmatrix} 0.5 & 0.8 \\ 0.5 & 0.2 \end{pmatrix}$ find $\mathbf{P}^2$.

$$\mathbf{P}^2 = \begin{pmatrix} 0.5 & 0.8 \\ 0.5 & 0.2 \end{pmatrix} \times \begin{pmatrix} 0.5 & 0.8 \\ 0.5 & 0.2 \end{pmatrix} = \begin{pmatrix} 0.25+0.4 & 0.4+0.16 \\ 0.25+0.1 & 0.4+0.04 \end{pmatrix}$$

$$= \begin{pmatrix} 0.65 & 0.56 \\ 0.35 & 0.44 \end{pmatrix}$$

A weather forecaster classifies the weather as dry or wet.
If it is wet on one day then the probability that it is wet the next day is 0.7. If it is dry one day then the probability that it is dry the next is 0.8.
(i) Form the transition matrix.
(ii) If it is dry one Monday what is the probability that it will be wet on Wednesday?

(i)

$$\mathbf{P} = \begin{array}{c} \\ \text{wet} \\ \text{dry} \end{array} \begin{array}{c} \text{wet} \quad \text{dry} \\ \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \end{array}.$$

(ii) The state for Monday is $\mathbf{M} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The state for Tuesday is $\mathbf{T} = \mathbf{PM}$

The state for Wednesday is $\mathbf{W} = \mathbf{PT} = \mathbf{P}^2\mathbf{M}$

$$\mathbf{W} = \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.55 & 0.3 \\ 0.45 & 0.7 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$$

so the probability that it will be wet on Wednesday is 0.3.

References: Chapter 5 Pages 177-179

3×3 Transition Matrices

If at any stage there are three states, then the transition matrix **P** will be a 3×3 matrix. There will be 9 transition probabilities.
The calculation of the product of these matrices (and those which are larger!) can be tedious so it is important that you are able to use your calculator effectively.
Usually (but not always) the transition matrix from stage 1 to stage 2 is the same as that from stage 2 to stage 3.
The transition **matrix** from stage 1 to stage 3 is therefore $\mathbf{P}^2$. You may therefore be required to calculate $\mathbf{P}^n$ for any integer value, *n*.

Example 5.1 Page 177

Exercise 5B Q. 1(i), 3

References: Chapter 5 Pages 182-187

Equilibrium probabilities

If, for some given starting state, successive states converge to fixed probabilities then those values are called **equilibrium probabilities**.
This means that at a limiting stage which gives a probability vector **p** then **Pp** = **p**.

This may occur in two situations:
(i) Whatever the initial column probability this stage is eventually reached.
(ii) If the initial column probability is the equilibrium probabilities then this state will be constant at all stages. This column probability vector can be found as follows

If $\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$ then $\mathbf{Pp} = \mathbf{p}$ gives

$$ap_1 + bp_2 = p_1$$
$$cp_1 + dp_2 = p_2$$

These can be solved simulataneously to find p_1 and p_2.

Exercise 5C Q. 2, 6

E.g. Find the Equilibrium probabilities for the matrix above.

Solve $\begin{pmatrix} 0.7 & 0.2 \\ 0.3 & 0.8 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$$0.7x + 0.2y = x \Rightarrow 0.2y = 0.3x \Rightarrow y = \frac{3}{2}x$$

Also $x + y = 1 \Rightarrow x + \frac{3}{2}x = 1 \Rightarrow x = \frac{2}{5} \Rightarrow y = \frac{3}{5}$

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References:
Chapter 5
Pages 190-192

Example 5.2
Page 192

Exercise 5D
Q. 1(i),(iii), 3

Run lengths in Markov Chains

The **run length** is the number of "no change" transitions. This is one less than the number of times the state is repeated.

So for any state A, if p is the probability that the system remains in that state at the next stage. Hence the probability that it changes from state A to state A′ is $1 - p$.

Let X represent the number of further consecutive stages in which the state of the system is A, given that it is initially in state A then

$$P(X = r) = p^r \times (1 - p) \quad \text{for } r = 0, 1, 2, 3, 4, ...$$

The expected run length is given by

$$E(X) = \sum[r \times P(X = r)] = p(1-p) + 2p^2(1-p) + 3p^3(1-p) + ...$$

$$= p(1-p)\left(1 + 2p + 3p^2 + ...\right)$$

$$= p(1-p) \times \frac{1}{(1-p)^2} = \frac{p}{1-p}$$

E.g. Expected run length of wet days for example on previous page.

$$\text{Length} = \frac{p}{1-p} = \frac{0.7}{1-0.7} = 2\frac{1}{3}$$

References:
Chapter 5
Pages 196-203

Example 5.3
Page 198

Example 5.4
Page 199

Exercise 5E
Q. 1, 3

Classifying Markov Chains

Regular chains

A transition matrix is *regular* if some power of the matrix has only positive entries. A Markov Chain is regular if its transition matrix is regular.

In a regular Markov chain it is possible to pass from any state to any other state and there is a unique limiting probability vector.

Random Walks

This is an expression that describes a process of moving between ordered states.

Periodic chains

A periodic Markov chain is one where successive powers of **P** form a pattern where there is a value of k such that $\mathbf{P}^k = \mathbf{P}$.

The **period** of the Markov chain is $k - 1$ where k is the smallest value for which this is true.

Reflecting barriers

A Markov chain has a *reflecting barrier* if following one particular state, the next state is inevitable. In the corresponding column of the transition matrix there is 0 in each position including position (i,i) except 1 which has the entry 1.

Absorbing states

A Markov chain has an *absorbing state* if the system is unable to leave that state once it has reached it. In the corresponding column of the transition matrix there is an entry of 1 in position (i,i) and 0 elsewhere.

E.g. The following matrix represents the transition matrix for the purchase by customers of three brands of a commodity, A, B and C.

$$\begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} \frac{3}{4} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{8} & \frac{1}{2} & \frac{1}{6} \\ \frac{1}{8} & \frac{1}{6} & \frac{1}{2} \end{pmatrix} \end{array}$$

(i.e. If a cutomer buys brand A then the chance of him buying it again is $\frac{3}{4}$; otherwise there is an equal chance of buying B or C.)

(i) Find the equilibrium probabilities.
(i.e. the long-run proportion of purchases.)

(ii) A customer buys brand A. Find the expected number of consecutive further occasions on which this customer purchased brand A.

(i) **PX = X** $\Rightarrow \frac{3}{4}x + \frac{1}{3}y + \frac{1}{3}z = x$

$$\frac{1}{8}x + \frac{1}{2}y + \frac{1}{6}z = y$$

$$\Rightarrow x = \frac{4}{3}(y + z) \quad \text{with } x + y + z = 1$$

$$\Rightarrow x = \frac{4}{7}, y = z = \frac{3}{14}$$

(ii) Expected run of purchaes of $A = \frac{p}{1-p}$ where p is the probability that A is purchased again after A

$$= \frac{\frac{3}{4}}{1 - \frac{3}{4}} = 3$$

E.g. The following matrix has an absorbing state.

$$\mathbf{P} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{pmatrix} 1 & 0.5 & 0.8 \\ 0 & 0.3 & 0.1 \\ 0 & 0.2 & 0.1 \end{pmatrix} \end{array}$$

If the start is at A then it will remain there.

On further steps, B will transfer to A with probability 0.5, B with probability 0.3 and C with probability 0.2.

These probabilities will reduce to 0, so A is an absorbing state.