REVISION SHEET – MECHANICS 2

CENTRE OF MASS (CoM)

The main ideas are

	AQA	Edx	MEI	OCR
Uniform bodies	M2	M2	M2	M2
Composite bodies	M2	M2	M2	M2
CoM when suspended	M2	M2	M2	-

Before the exam you should know:

- How to find the centre of mass of a system of particles of given position and mass.
- How to locate centre of mass by appeal to symmetry.
- How to find the centre of a mass of a composite body by considering each constituent part as a particle at its centre of mass.
- How to use the position of the centre of mass in problems involving the equilibrium of a rigid body.

Centre of Mass

In much of the work that you have done involving moments, the whole weight of a rigid body, such as a rod, was considered to act at its centre. This is called the centre of mass. However, if the rod did not have uniform distribution of its mass, then the centre of mass would be at a different location.

Uniform Bodies

For a uniform body:

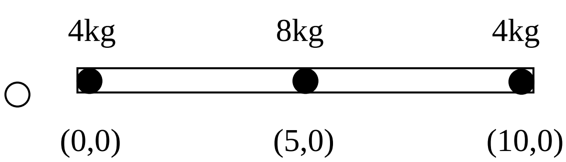
Moment of the whole mass at the centre of mass = sum of the moments of the individual masses

i.e. $(\sum m)\bar{x} = \sum (mx)$

A standard application of this method can be seen in the example to the right.

In examples using CoM it is always useful to check whether there is any symmetry present. (Although note that you should be careful to check that symmetry is actually present rather than assuming there is!)

For instance, in the example below, which is similar to the question to the right, it is possible to say that, by symmetry, the CoM will be at (5, 0).

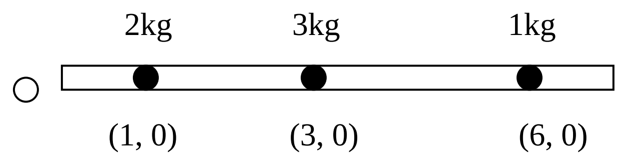


Example *(CoM)*
A rod A, of length 3 metres, has uniform mass. A rod B, also of length 3 metres, does not have uniform mass, with its mass concentrated towards one of its ends. For each rod, say whether it is possible to give the position of their CoM using the information give, If it is possible, state where the CoM is.

It can be clearly seen that for rod A the CoM acts half way along the rod, i.e. 1.5 m. However, for rod B, without further information you are unable to say where its CoM acts, except that because it is not uniformly distributed it will not lie in the centre.

Example *(Uniform bodies)*

Particles of mass 2kg, 3kg and 1kg are at the points (1, 0), (3, 0) and (6, 0) on a light rod which lies along the *x*-axis. What are coordinates of the centre of mass?



Relative to O:

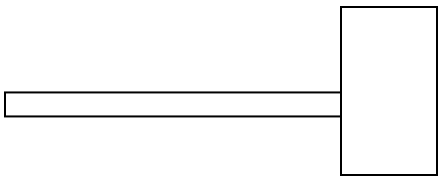
$$(\sum m)\bar{x} = \sum (mx)$$
$$(2+3+1)\bar{x} = (2\times 1) + (3\times 3) + (1\times 6)$$
$$6\bar{x} = 15$$
$$\bar{x} = 15/6 = 5/2$$
$$= 2.5$$

Therefore, the centre of mass lies at (2.5, 0).

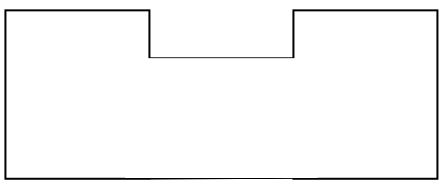
Composite Bodies

In the case for composite bodies you need to consider how 'individual bodies' can be put together to form the 'whole body'.

For instance a hammer may be represented by combining two bodies and considering the CoM of each of them:



A lamina like the one below could be considered as a number of separate bodies in several ways, i.e. one large rectangle (the whole length) plus the two smaller ones above, or three rectangles created by separating the three 'vertical' parts.



CoM when suspended *(not required for OCR)*

Within questions on CoM, once the actual CoM has been calculated then you may be asked to consider the body suspended from a specific point.

The key to these questions often lies in having a good, large clear diagram as in order to find the answer trigonometry is needed.

In the example to the right the lamina was turned so that when it was suspended from A, the line between A and the CoM was vertical. When doing this by hand it is better NOT to try and draw a whole new diagram, but simply to draw the line between the CoM and the relevant point. Once this line has been drawn careful consideration of the lengths and angles needs to be given, so that the required angle can be found.

Example *(Composite bodies)*

Find the coordinates of the centre of mass of five particles of mass 5kg, 4kg, 3kg, 2kg and 1kg situated at (4, 4), (7, 7) (1, 1), (7, 1) and (1, 7) respectively.

In order to solve this consider the x and y coordinates separately.

For the x coordinate:

$$(\sum m)\overline{x} = \sum(mx)$$

$$(5+4+3+2+1)\,\overline{x} = (5\times4)+(4\times7)+(3\times1)+(2\times7)+(1\times1)$$

$$15\overline{x} = 66$$

$$\overline{x} = {}^{66}\!/\!_{15} = 4.4$$

For the y coordinate:

$$(\sum m)\overline{y} = \sum(my)$$

$$(5+4+3+2+1)\,\overline{x} = (5\times4)+(4\times7)+(3\times1)+(2\times1)+(1\times7)$$

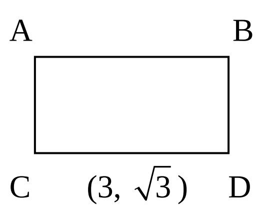
$$15\overline{y} = 60$$

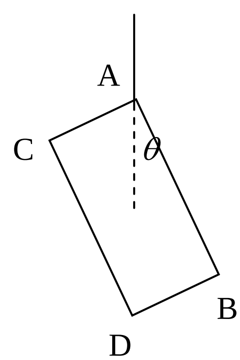
$$\overline{y} = {}^{60}\!/\!_{15} = 4$$

Therefore, the centre of mass lies at (4.4, 4).

Examples *(CoM when suspended)*

A regular lamina ABCD, whose CoM is at the point $(3, \sqrt{3})$, is suspended from point A. What angle does AB make with the vertical?





By trigonometry,

$$\tan\theta = \frac{\sqrt{3}}{3}$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{3} = \frac{\pi}{6}(\approx 0.52)$$

REVISION SHEET – MECHANICS 2

CIRCULAR MOTION

The main ideas are

	AQA	Edx	MEI	OCR
Angular velocity	M2	M3	M3	M2
Acceleration	M2	M3	M3	M2
Horizontal and vertical circles	M2	M3	M3	M2

Before the exam you should know:

- How to calculate angular velocities.
- The formulae for radial and transverse acceleration and how to apply them.
- How to solve problems in horizontal and vertical circles, including the use of energy.

Angular velocity

For a particle moving in a circle, or radius *r*, the relationship between the actual speed *v* and the angular velocity ω is:

$$v = r\omega$$

This is a relatively simple formulae to use but you need to be careful that the units are consistent. They may be in rad s^{-1} or if it is revolutions per sec (rpm) you may wish to convert it to rad s^{-1}.

Acceleration

The acceleration towards the centre of a circle, known as the radial acceleration, of a particle moving with angular velocity ω in a circle of radius *r* is given by:

$$a = r\omega^2$$

Using the relationship between speed and angular velocity gives the alternative formula:

$$a = \frac{v^2}{r} \quad (\text{or } r\dot{\theta}^2)$$

The acceleration tangential (or transverse) to the centre of a circle is of the standard form:

$$a = \frac{dv}{dt} \, (\text{or } r\dot{\omega} \text{ or } r\ddot{\theta})$$

These formulae are used in conjunction with Newton's laws to find forces. The forces may subsequently be needed to find relationships using energy principles with respect to given circular motion situations (particularly vertical circles), see the text on the second page.

Horizontal and vertical circles

Example *(Angular velocity)*

A particle is travelling in a circle, of radius 0.25 m, with angular velocity of 6 rad s^{-1}. What is the particle's speed and what is the particle's angular velocity in rpm?

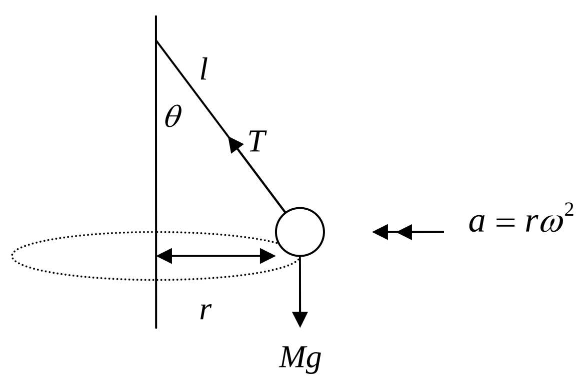
$v = r\omega = 0.25 \times 6 = 1.5$ m s^{-1}

In order to convert from rad s^{-1} to rpm, need to multiply the rad s^{-1} by $60/2\pi$,

i.e. $6 \times 60/2\pi$ rad s^{-1} = 57.30 rpm

Example *(Acceleration)*

A conical pendulum of length *l* with a bob, of mass *M*, is rotating in a horizontal circle of radius *r*. Draw a force diagram to show the motion. What are the modelling assumptions used?



The modelling assumptions are that the bob will be modelled as a particle, which is attached to a light inextensible string and that air resistance is ignored.

Within this unit you will need to use circular motion in the context of both horizontal and vertical circles. There are many situations that can be modelled using these.

Horizontal circles

In horizontal circular motion, as with forces on a plane in general, it is important to consider the direction in which you resolve forces. You are likely to still need to resolve either parallel and/ or perpendicular to the plane, but now you also need to be able to interpret acceleration towards the centre of a circle. See the example to the right which describes a particle on a banked surface.

Vertical circles

Many different situations arise involving vertical motion and each have their own conditions. Some typical ones include:

A particle on a string: Here you are likely to use the idea that when the string is not taut the tension will be taken to be 0. Also questions could involve energy and require calculations for initial or final speeds.

An object (ring, bead) threaded onto a wire/ string: Here the reaction between the two, i.e. string and ring needs to be considered.

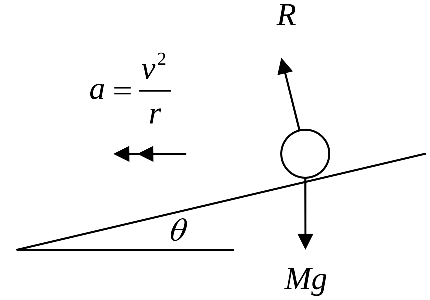
A particle on the outside of a vertical circle: The crucial point is that when the particle 'leaves' the surface the normal reaction will be 0.

A particle on the inside of a vertical circle: Here you may be asked if the particle 'completes' a vertical circle or if it leaves the surface, again this will involve considering energy, the speed of the particle and the normal reaction.

As can be seen there are a number of different situations that can be modelled using vertical and horizontal circles so it is wise to review each of the possibilities and note the key features. The calculations may be very similar but are dependant on one or two key 'assumptions'.

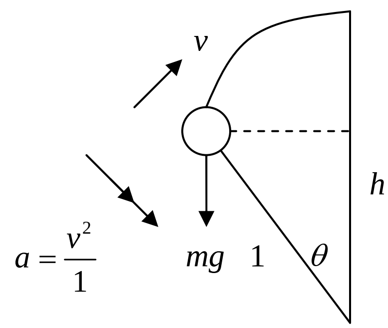
Examples *(Horizontal and vertical circles)*

1. A car, of mass m is travelling around a bend, which is banked at an angle of θ^{o}. Draw a diagram to model the situation. Is the speed at which it is travelling, to ensure it does not slip down the corner, dependent on the car's mass?



The speed is not dependant on the car's mass, can you show this by working through the calculations?

2. A smooth circular wire, of radius 1, is fixed in a vertical plane. A small ring, of mass m, is free to move on the wire. If the ring is projected upwards, from the horizontal, with speed, 1 m s^{-1}, find an expression for the height, h, above the horizontal in terms of its speed, v, when the normal reaction is 0.



When R = 0, using Newton's second law

$$mg\cos\theta = m(v^2/1) \quad \ldots(1)$$

Here, $\cos\theta = (h/1) = h$

So (1) becomes: $v^2 = gh$

h can be found by considering cons. of energy

loss in KE = gain in GPE

$$\frac{1}{2}m(1)^2 - \frac{1}{2}mv^2 = mgh$$

Substituting in $v^2 = gh$ and rearranging gives:

$$h = 2/3$$

REVISION SHEET – MECHANICS 2

ENERGY, WORK & POWER

The main ideas are

	AQA	Edx	MEI	OCR
Work done	M2	M2	M2	M2
Kinetic energy	M2	M2	M2	M2
Gravitational potential energy	M2	M2	M2	M2
Power	M2	M2	M2	M2

Before the exam you should know:

- How to calculate the work done by a force.
- How to calculate kinetic energy and gravitational potential energy.
- The term mechanical energy and the work-energy principle.
- Be able to solve problems using the principle of conservation of energy.
- How to apply the concept of power to the solution of problems.

Work done

The work done by a constant force = force x distance moved in the direction of the force.
i.e. work done = Fs, where s is the distance. (Note: Sometimes you may see d used instead of s.)

However, if the displacement is not parallel to the force, then the force will need to be resolved to find the parallel component. i.e. if the force F acts at an angle θ to the parallel the Work done will equal $Fd\cos\theta$.

Kinetic energy

The energy possessed by a body because of its speed:

$$\text{Kinetic energy} = \frac{1}{2} \times \text{mass} \times (\text{speed})^2$$
$$= \frac{1}{2}mv^2$$

Also, in some questions it can be useful to use the idea that:
work done by a force = final KE – initial KE

Gravitational potential energy (GPE)

GPE is just one form of potential energy. You can think of potential energy as energy stored in an object, which gives it the potential to move when released. If a ball is dropped, the gravitational potential energy of the ball is converted into kinetic energy. As the height of the object decreases, the gravitational potential of the ball decreases, and its kinetic energy increases.

Example *(Work done)*
A bus on level ground is subject to a resisting force (from its brakes) of 10 kN for a distance of 400 m. How much energy does the bus lose?

The forward force is –10 000N.
The work done by it is –10 000 x 400
= –4 000 000 J
Hence –4 000 000 J of energy is lost and the bus slows down.

Example *(Kinetic energy)*
A bullet, of mass 30 grams, is fired at a wooden barrier 2.5 cm thick. When it hits the barrier it is travelling at 200 ms^{-1}. The barrier exerts a constant resistive force of 4500 N on the bullet. Does the bullet pass through the barrier and if so with what speed does it emerge?

$$\text{Work done} = -4500 \times 0.025\,\text{J} = -112.5\text{ J}$$
(*note negative as work done by the force has to be defined in the direction of the force*)

$$\text{Initial KE} = \frac{1}{2}mu^2 = 0.5 \times 0.03 \times 200^2 = 600\text{ J}$$

A loss of 112.5 J will not reduce the KE to below 0 so the bullet will still be moving on exit.

As the work done is equal to the change in kinetic energy,

$$-112.5 = \frac{1}{2}mv^2 - 600$$

(example continues on the next page)

Elastic potential energy (EPE) and Hooke's Law **(AQA ONLY)**

Another form of potential energy, which AQA students require in M2, is elastic potential energy. This is the energy stored in a spring when it is stretched or compressed. $EPE = \frac{\lambda x^2}{2l}$. An object attached to the spring has the potential to move when the spring is released. This is often used in conjunction with Hooke's Law. The law is used in the form $T = kx$ where k is the stiffness of the string/ spring, or in the form $T = \frac{\lambda x}{l}$, where l is the natural length and λ is the modulus of elasticity. Please refer to your notes and recommended text for example questions.

Energy

Kinetic energy and gravitational energy are both forms of mechanical energy. When gravity is the only force acting on a body, total mechanical energy is always conserved.

When solving problems involving a change in vertical position, it is often convenient to use the work-energy principle (the total work done by the forces acting on a body is equal to the increase itn he kinetic energy of the body) in a slightly different form.

That is:
Work done by external forces other than weight
= *mgh* + increase in KE
= increase in GPE + in crease in KE
= increase in total mechanical energy

Power

Power is the rate of doing work

The power of a vehicle moving at speed *v* under a driving force *F* is given by *Fv*. Power is measured in Watts (W).

For a motor vehicle it is the engine which produces the power, whereas for a cyclist riding a bike it is the cyclist.

Solving for *v*

$$\frac{1}{2}mv^2 = 600 - 112.5$$

$$v^2 = \frac{2(600-112.5)}{0.03}$$

$$v = 180 \text{ ms}^{-1}$$

So the bullet will exit with a speed of 180 ms^{-1}

Example *(GPE)*

Calculate the gravitational potential energy, relative to the ground, of a small ball of mass 0.2 kg at a height of 1.6 m above the ground.

Mass m = 0.2, height h = 1.6

GPE = mgh $= 0.2 \times 9.8 \times 1.6$
$= 3.14$ J

If the ball falls then:
Loss in PE = work done by gravity = gain in KE
(There is no change in the total energy (KE + PE) of the ball.)

Note. The examples here could be classified as simple examples. It should be recognised that examination questions on Energy are likely to be much more involved, however, due to the space limitation here that level of example could not be posed. Please see your notes and recommended text for further examples

Example *(Power)*

A car of mass 1100 kg can produce a maximum power of 44000 W. The driver of the car wishes to overtake another vehicle. If air resistance is ignored, what is the maximum acceleration of the car when it is travelling at 30 ms^{-1} ?

Power = force x velocity
The driving force at 30 ms^{-1} is F N, where
44000 = F x 30

$$F = \frac{44000}{30} = \frac{4400}{3} \approx 1466.66 \text{ N}$$

By Newton's second law $F = ma$

$$\text{acceleration} = \frac{1466.66}{1100} = \frac{4}{3} \approx 1.33 \text{ ms}^{-2}$$

REVISION SHEET – MECHANICS 2

EQUILIBRIUM OF A RIGID BODY

The main ideas are

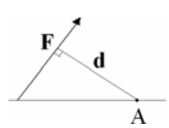
	AQA	Edx	MEI	OCR
Moments	M2	M2	M2	M2

Before the exam you should know:

- What moments are and how to calculate them.

Moments (moment of a force)

The moment of the force, **F**, about an axis through A, perpendicular to the plane containing A and **F**, is *Fd*.



Where there is a hinge or a fulcrum there is always some kind of reaction force at the hinge or fulcrum. This is why it often makes sense to take moments about a hinge or a fulcrum, as the reaction has no moment about that point.

Remember to use the principle that, under the action of coplanar forces, a rigid body is in equilibrium if and only if: the vector sum of the forces is zero, and the sum of the moments of the forces about any point is zero.

Many problems can be solved by using a combination of resolving forces and taking moments.

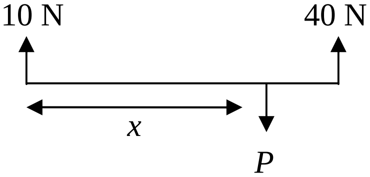
Always draw a diagram **–** if you try to work without a diagram, you are very likely to make mistakes with signs, or to miss out forces.

Remember to include reaction forces at a support or hinge in the force diagram. These have no effect when you take moments about the support or hinge, but you need to take them into account when you resolve forces or take moments about a different point.

Practice the more difficult examples which have many forces involved, e.g. a ladder on a rough surface placed against a rough wall etc.

Examples *(Moments)*

1. What is the distance, *x*, required for a light horizontal rod, of length 1.5 m to be resting in equilibrium?

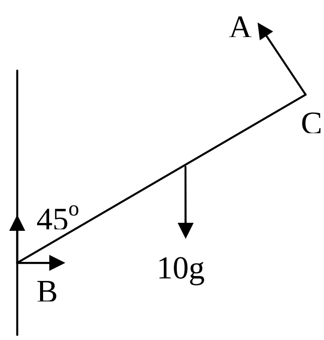


Resolving vertically: $10 + 40 - P = 0, \Rightarrow P = 50$ N

Taking moments about left-hand end:

$$40 \times 1.5 - Px = 0$$
$$50x = 60$$
$$x = 6/5 = 1.2 \text{ m}$$

2. A uniform rod of mass 10g is pivoted at B and held at an angle of 45^o to the vertical by a force applied at C, perpendicular to BC. What is the force A?



If the length of the rod is said to be 2*x*, then taking moments about B (as both the forces at B go through that point)

$$10gx\cos 45^o = A \times 2x$$

$$A = 10g\cos 45^o / 2 = 34.65\,\text{N}$$

REVISION SHEET – MECHANICS 2

VARIABLE ACCELERATION USING DIFFERENTIATION AND INTEGRATION

The main ideas are

	AQA	Edx	MEI	OCR
Differentiation	M2	M2	M1	M1
Integration	M2	M2	M1	M1
Differentiation in 2 dimensions	M2	M2	M1	M1
Integration in 2 dimensions	M2	M2	M1	M1

The main ideas in this topic are:

- **Using differentiation and integration to obtain expressions for the displacement, velocity and acceleration from one another. You should be able to do this in one, two or three dimensions.**
- **Obtaining values of associated quantities such as speed and distance travelled.**

Before the exam you should know:

- **Velocity is the rate of change of displacement.** Therefore to obtain an expression for a particle's velocity at time *t* you should differentiate the expression for its displacement.
- **Acceleration is the rate of change of velocity.** Therefore to obtain an expression for a particle's acceleration at time *t* you should differentiate the expression for its velocity.
- Reversing the two ideas above, **a particle's velocity can be obtained by integrating the expression for its acceleration** and **a particle's displacement can be obtained by integrating the expression for its velocity**. In both cases this will introduce a constant of integration whose value can be found if the particle's displacement or velocity is known at some particular time.
- The above facts apply to both:
  1. particles travelling in one dimension. In this case each of the displacement, velocity and acceleration is a (scalar valued) function of time, all of which can be differentiated and integrated in the usual way.
  2. particles travelling in two and three dimensions, when the displacement, velocity and acceleration are all vectors with components dependent on *t* (time). We differentiate and integrate such expressions in the usual way, dealing with each component separately. There are several examples of this on this sheet.
- You should be comfortable with both column vector and **i**, **j**, **k** notation for vectors.
- (*For AQA and Edexcel*) How to solve a differential equation by separating the variables and then solve using integration techniques. (please refer to your notes on this topic)

Using differentiation - a particle travelling in a straight line.

***Example**: An object moves in a straight line, so that its displacement relative to some fixed origin at time t is given by* $s = t^3 - 5t^2 + 4$.

1. *Find expressions for its velocity and acceleration at time t.*
2. *Calculate the velocity and acceleration of the object when t = 0 and when t = 1.*
3. *What is the displacement of the object when its velocity is zero?*

Solution.

1. $s = t^3 - 5t^2 + 4$ so that $v = \frac{ds}{dt} = 3t^2 - 10t$ and $a = \frac{dv}{dt} = 6t - 10$.
2. When $t = 0, v = 3\times0 - 10\times0 = 0\text{ms}^{-1}$ and $a = 6\times0 - 10 = -10\text{ms}^{-2}$ and when $t = 1, v = 3\times1 - 10\times1 = -7\text{ms}^{-1}$ and $a = 6\times1 - 10 = -4\text{ms}^{-2}$.
3. The velocity of the object at time t is $3t^2 - 10t = t(3t - 10)$. This is zero when t = 0 or when $t = \frac{10}{3}$. The displacement of the particle when t = 0 is $s = 4\text{m}$ and then displacement of the particle when $t = \frac{10}{3}$ is

Using integration – a particle travelling in a straight line.

Example*: An object is moving in a straight line with acceleration at time t given by* $a = 10 - 6t$.
Given that when t = 1, s = 0 and v = –5, where s is the object's displacement and v is the object's velocity, find an expression for v and s in terms of t.
Hence find out the displacement of the particle when it first comes to rest.

Solution

$v = \int a \; dt = \int (10 - 6t) dt = 10t - 3t^2 + c$. But when t = 1, $v = 5 = 10 - 3 + c \Rightarrow c = -2$. So $v = 10t - 3t^2 - 2$.

$s = \int v \; dt = \int (10t - 3t^2 - 2) \; dt = 5t^2 - t^3 - 2t + c$. But when t = 1, $s = 0 = 5 - 1 - 2 + c \Rightarrow c = -2$. So $s = 5t^2 - t^3 - 2t - 2$

Using differentiation – an example in two dimensions using column vector notation.

Example*: A girl throws a ball and, t seconds after she releases it, its position in metres relative to the point where she is standing is modelled by*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15t \\ 2 + 16t - 5t^2 \end{pmatrix}$$

where the directions are horizontal and vertical.

1. *Find expressions for the velocity and acceleration of the ball at time t.*
2. *The vertical component of the velocity is zero when the ball is at its highest point. Find the time taken for the ball to reach this point.*
3. *What is the speed of the ball when it hits the ground.*

Solution

1. The velocity is obtained by differentiating (with respect to t) the components in the vector giving the ball's position. This gives $v = \begin{pmatrix} 15 \\ 16 - 10t \end{pmatrix}$. The acceleration is obtained by differentiating (with respect to t) the components in the vector giving the ball's velocity. This gives $a = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$.
2. The vertical component of the velocity is 16 – 10t. This is zero when $t = \frac{16}{10} = \frac{8}{5} = 1.6 \text{ seconds}$.
3. The ball hits the ground when the vertical component of the balls position is zero. In other words when $2 + 16t - 5t^2 = 0$. Rearranging this as $5t^2 - 16t - 2 = 0$ and then using the formula for the solutions of a quadratic we see that the solutions of this are $t = -0.12$ and $t = 3.3$ (to 2 s.f). Clearly the value we require is t = 3.3. The velocity of the ball when t = 3.3 is $\begin{pmatrix} 15 \\ -17 \end{pmatrix}$ and so the speed is $\sqrt{15^2 + (-17)^2} = 22.67 \text{ ms}^{-1}$.

Using Integration and Newtons 2nd Law an example in 2 dimensions with i, j notation.

Example*: A particle of mass 0.5 kg is acted on by a force, in Newtons, of* $\mathbf{F} = t^2\mathbf{i} + 2t\mathbf{j}$. *The particle is initially at rest and t is measured in seconds.*

1. *Find the acceleration of the particle at time t.*
2. *Find the velocity of the particle at time t.*

Solution

Newton's second law, $\mathbf{F} = m\mathbf{a}$ gives that $\mathbf{F} = t^2\mathbf{i} + 2t\mathbf{j} = 0.5\mathbf{a}$ so that $\mathbf{a} = 2t^2\mathbf{i} + 4t\mathbf{j}$.

We have that $\mathbf{v} = \int \mathbf{a} \; dt = \left(\frac{2t^3}{3} + c \right)\mathbf{i} + \left(2t^2 + d \right)\mathbf{j}$ where c and d are the so-called "constants of integration".

We are told that the particle is at rest when t = 0 and so $c = d = 0$. This gives $\mathbf{v} = \frac{2t^3}{3}\mathbf{i} + 2t^2\mathbf{j}$.