Mechanics 2

Revision Notes

October 2016

Mechanics 2

1 Kinematics

Constant acceleration in a vertical plane

We can think of the horizontal and vertical motion separately and use the formulae:

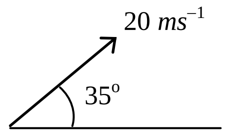
$v = u + at$, $\quad s = ut + \frac{1}{2}at^2$, $\quad v^2 = u^2 + 2as$, $\quad s = \frac{1}{2}(u + v)t$

Example 1: A stone is thrown at a speed of 20 ms^{-1} at an angle of 35° to the horizontal.

Find (*a*) the greatest height reached
(*b*) the direction in which it is moving after 1 second
(c) the height of the stone after it has travelled a horizontal distance of 25 *m*.

Solution:

(*a*) Vertical motion $\uparrow +$



$u = 20 \sin 35^\circ$, $\quad a = -9.8$, $\quad v = 0$, $\quad s = h$

$v^2 = u^2 + 2as$

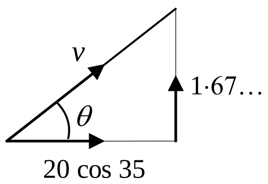
$\Rightarrow \quad 0 = (20 \sin 35^\circ)^2 - 2 \times 9{\cdot}8h$

$\Rightarrow \quad h = 6{\cdot}71408017$

$\Rightarrow$ greatest height reached is $6{\cdot}7\ m$ to 2 S.F.

(*b*) Horizontal motion $\rightarrow +$ $\qquad u = 20 \cos 35^\circ$

Vertical motion $\uparrow +$ $\qquad u = 20 \sin 35^\circ$, $\quad a = -9.8$, $\quad t = 1$, $v = v_v$



$v_v = u + at = 20 \sin 35^\circ - 9{\cdot}8 \times 1 = 1{\cdot}671528727$

$\Rightarrow \quad \tan\theta = \frac{1{\cdot}671\ldots}{20 \cos 35}$

$\Rightarrow \quad \theta = 5{\cdot}8256\ldots$

$\Rightarrow$ stone is moving at $5{\cdot}8^\circ$ **above** the horizontal, 1 D.P.

(*c*) Horizontal motion $\rightarrow +$ $\qquad u = 20 \cos 35^\circ$, $\quad s = 25$, $\quad a = 0$

$$\Rightarrow \qquad t = \frac{25}{20 \cos 35} = 1 \cdot 525968236$$

Vertical motion $\uparrow +$ $\qquad u = 20 \sin 35^\circ$, $\quad a = -9.8$, $\quad t = 1{\cdot}5259\ldots$, $\quad s = ?$

$s = ut + \frac{1}{2}at^2 = 20 \sin 35^\circ \times 1{\cdot}5259\ldots - \frac{1}{2} \times 9{\cdot}8 \times (1{\cdot}525\ldots)^2$

$\Rightarrow \qquad s = 6{\cdot}095151075$

$\Rightarrow$ stone is at a height of $6{\cdot}1\ m$ after it has travelled 25 *m* horizontally 2 S.F.

Example 2: A ball is projected from a point on horizontal ground with a speed of U, making an angle of α with the horizontal.

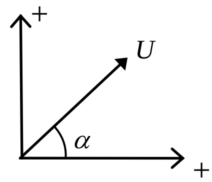
(*a*) Show that the ball moves along a parabola.

(*b*) Find the range.

(c) If α is allowed to vary, find the *maximum* range.

Solution: Take the point of projection as the origin, and let the particle be at the point (x, y) at time T.

(*a*) Horizontal motion $\rightarrow +$



$u = U\cos\alpha, \quad s = x, \quad t = T$

$\Rightarrow \quad x = (U\cos\alpha)\,T$

$\Rightarrow \quad T = \dfrac{x}{U\cos\alpha}$ **I**

Vertical motion $\uparrow +$

$u = U\sin\alpha, \quad s = y, \quad a = -g, \quad t = T$ note that T is the same for both directions

$s = ut + \frac{1}{2}at^2 \quad \Rightarrow \quad y = (U\sin\alpha)\,T - \frac{1}{2}g\,T^2$ **II**

I and **II** $\Rightarrow \quad y = (U\sin\alpha)\left(\dfrac{x}{U\cos\alpha}\right) - \dfrac{1}{2}g\left(\dfrac{x}{U\cos\alpha}\right)^2$

$\Rightarrow \quad y = x\tan\alpha - \dfrac{g\sec^2\alpha}{2U^2}x^2$

which is a quadratic function of x, and so the ball moves in a parabola.

(*b*) When the ball hits the ground, $y = 0$

$\Rightarrow$ range is solution of $0 = x\tan\alpha - \dfrac{g\sec^2\alpha}{2U^2}x^2$

$\Rightarrow \quad x\left(\tan\alpha - \dfrac{g\sec^2\alpha}{2U^2}x\right) = 0$

$\Rightarrow \quad x = 0$ (the start), or $x = \dfrac{2U^2\tan\alpha}{g\sec^2\alpha} = \dfrac{2U^2\sin\alpha\cos\alpha}{g}$

$\Rightarrow$ range is $\dfrac{2U^2\sin\alpha\cos\alpha}{g}$

(*c*) The range is $\dfrac{2U^2\sin\alpha\cos\alpha}{g} = \dfrac{U^2\sin 2\alpha}{g}$

and, since the maximum value of $\sin\theta$ is 1, the maximum range is

$\dfrac{U^2}{g}$, and occurs when $\alpha = 45^\circ$.

Variable acceleration

When a is given as a function of t (**not** constant)

$v = \int a \; dt$ do not forget the $+c$

$s = \int v \; dt$ do not forget the $+c$

When s (or v) is given as a function of t

$v = \frac{ds}{dt}$

and $a = \frac{dv}{dt}$ or $a = \frac{d^2s}{dt^2}$

Note that s is the **displacement** (the distance from the origin), which is **not** necessarily the same as the distance travelled (the particle may have moved forwards and backwards).

Example 1: A particle is moving along the x-axis with an acceleration $5 - 2t \; ms^{-2}$. At time $t = 0$, the particle moves through the origin with speed $6 \; ms^{-1}$ in the direction of the positive x-axis.

(a) Find the displacement of the particle after 9 seconds.

(b) Find the distance travelled in the first 9 seconds.

Solution:

(a) $a = \frac{dv}{dt} = 5 - 2t \Rightarrow \quad v = \int 5 - 2t \; dt = 5t - t^2 + c$

$v = 6$ when $t = 0$, $\Rightarrow c = 6$

$\Rightarrow \quad v = 5t - t^2 + 6$

$s = \int v \; dt = \int 6 + 5t - t^2 \; dt = 6t + \frac{5}{2}t^2 - \frac{1}{3}t^3 + c'$

$s = 0$ when $t = 0$, $\Rightarrow c' = 0$

$\Rightarrow \quad s = 6t + \frac{5}{2}t^2 - \frac{1}{3}t^3$

When $t = 9$, $s = 6 \times 9 + \frac{5}{2} \times 9^2 - \frac{1}{3} \times 9^3 = 13{\cdot}5$

The displacement after 9 seconds is $13{\cdot}5 \; m$.

(b) **Note:** *the particle could have gone forwards then backwards, in which case the distance travelled would not be the same as the final displacement.*

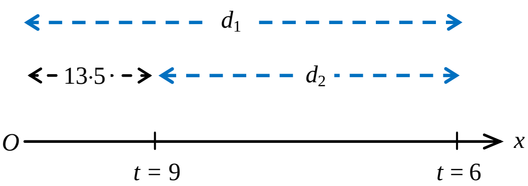
We must first find t when the velocity is zero.

$v = 5t - t^2 + 6 = (6 - t)(1 + t)$

$\Rightarrow \quad v = 0$ when $t = (-1)$ or 6.

The particle is moving away from the origin for $0 \le t < 6$, and towards the origin for $6 < t \;\; 9$, so we want the sum of the two distances $d_1 + d_2$.

When $t = 6$, $s = d_1$



$\Rightarrow \quad d_1 = 6 \times 6 + \frac{5}{2} \times 6^2 - \frac{1}{3} \times 6^3$

$\Rightarrow \quad d_1 = 54$

$\Rightarrow \quad d_2 = 54 - 13{\cdot}5 = 40{\cdot}5$

$\Rightarrow \quad$ total distance travelled $= d_1 + d_2 = 94{\cdot}5 \; m$

Using vectors

This is just combining horizontal and vertical motion into one expression.

Example 1: A particle moves with velocity $\underline{\boldsymbol{v}} = \begin{pmatrix} 3t^2 \\ -4t \end{pmatrix}$ ms^{-1}. It is initially at the point (6, 3).

Find (a) its acceleration after 2 seconds, and (b) its displacement at time t.

Solution: (a) $\underline{\boldsymbol{a}} = \frac{d\underline{\boldsymbol{v}}}{dt} = \frac{d\begin{pmatrix} 3t^2 \\ -4t \end{pmatrix}}{dt} = \begin{pmatrix} 6t \\ -4 \end{pmatrix}$

$\Rightarrow$ at $t = 2$, $\underline{\boldsymbol{a}} = \begin{pmatrix} 12 \\ -4 \end{pmatrix}$ ms^{-2}

(b) $\underline{\boldsymbol{s}} = \int \underline{\boldsymbol{v}}\ dt = \int \begin{pmatrix} 3t^2 \\ -4t \end{pmatrix} dt = \begin{pmatrix} t^3 + c_1 \\ -2t^2 + c_2 \end{pmatrix}$

Particle is initially, $t = 0$, at $\underline{\boldsymbol{s}} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$ $\Rightarrow$ $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix}$

$\Rightarrow$ $\underline{\boldsymbol{s}} = \begin{pmatrix} t^3 + 6 \\ -2t^2 + 3 \end{pmatrix}$ m

Example 2: A particle A is at (1, 2) when $t = 0$, moving with velocity $\begin{pmatrix} 4t \\ -3 \end{pmatrix}$ ms^{-1}. A second particle, B, is at the origin when $t = 0$, moving with velocity $\begin{pmatrix} 2t \\ -2t \end{pmatrix}$ ms^{-1}.

Investigate whether A and B collide.

Solution: For A $\underline{\boldsymbol{s_A}} = \int \begin{pmatrix} 4t \\ -3 \end{pmatrix} dt = \begin{pmatrix} 2t^2 \\ -3t \end{pmatrix} + \underline{\boldsymbol{c}}_1$

$\underline{\boldsymbol{s_A}} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ at $t = 0$ $\Rightarrow$ $\underline{\boldsymbol{c}}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\Rightarrow$ $\underline{\boldsymbol{s_A}} = \begin{pmatrix} 2t^2 + 1 \\ -3t + 2 \end{pmatrix}$

For B $\underline{\boldsymbol{s_B}} = \int \begin{pmatrix} 2t \\ -2t \end{pmatrix} dt = \begin{pmatrix} t^2 \\ -t^2 \end{pmatrix} + \underline{\boldsymbol{c}}_2$

$\underline{\boldsymbol{s_B}} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ at $t = 0$ $\Rightarrow$ $\underline{\boldsymbol{c}}_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow$ $\underline{\boldsymbol{s_B}} = \begin{pmatrix} t^2 \\ -t^2 \end{pmatrix}$

If they collide, both x and y coordinates must be equal for the ***same*** value of t.

The y coordinates are equal when $2 - 3t = -t^2$

$\Rightarrow$ $t^2 - 3t + 2 = 0$ $\Rightarrow$ $(t - 1)(t - 2) = 0$

$\Rightarrow$ $t = 1$ or 2.

When $t = 1$, $\underline{\boldsymbol{s_A}} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ and $\underline{\boldsymbol{s_B}} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\Rightarrow$ $\underline{\boldsymbol{s_A}} \neq \underline{\boldsymbol{s_B}}$

When $t = 2$, $\underline{\boldsymbol{s_A}} = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$ and $\underline{\boldsymbol{s_B}} = \begin{pmatrix} 4 \\ -4 \end{pmatrix}$ $\Rightarrow$ $\underline{\boldsymbol{s_A}} \neq \underline{\boldsymbol{s_B}}$

$\Rightarrow$ A and B do **not** collide.

2 Centres of mass

Centre of mass of *n* particles

The centre of mass, $(\bar{x}, \bar{y})$, of n particles,
which have masses $m_1, m_2, \ldots, m_n$, at points (x_1, y_1), (x_2, y_2), …, (x_n, y_n) is given by

$$M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \sum m_i \begin{pmatrix}x_i\\ y_i\end{pmatrix}$$

$\Leftrightarrow$ $\quad M\,\bar{x} = \sum m_i\, x_i$ and $M\,\bar{y} = \sum m_i\, y_i$

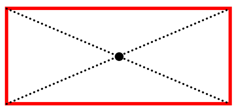
where $\boldsymbol{M}$ is the total mass, $\boldsymbol{M} = \sum m_i$

or $\quad M\,\underline{\bar{\boldsymbol{r}}} = \sum m_i\, \underline{\boldsymbol{r}}_i$

Centres of mass of simple laminas

1) The centre of mass of a *uniform* rod is at its mid point.

2) The centre of mass of a *uniform* rectangular lamina (sheet) is at its point of symmetry.

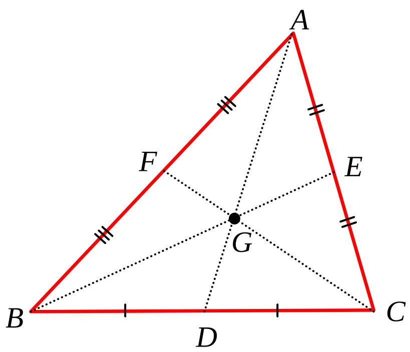


3) The centre of mass of a uniform triangular lamina

(*a*) G is at the point where the three medians meet.

G divides each median in the ratio 2:1

$$\frac{AG}{GD} = \frac{BG}{GE} = \frac{CG}{GF} = \frac{2}{1}$$



(*b*) If A, B and C are the points (a_1, a_2), (b_1, b_2) and (c_1, c_2), then the centre of mass, G, is at the point

$$G\ \left(\tfrac{1}{3}(a_1 + b_1 + c_1),\ \tfrac{1}{3}(a_2 + b_2 + c_2)\right)$$

or $\quad \underline{\bar{\boldsymbol{r}}} = \frac{1}{3}(\underline{\boldsymbol{a}} + \underline{\boldsymbol{b}} + \underline{\boldsymbol{c}})$

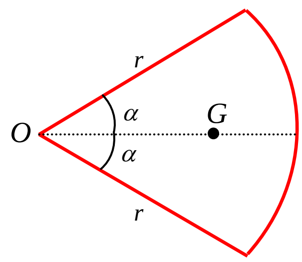
4) The centre of mass of a uniform *lamina* in the shape of a sector of a circle, with radius r and angle 2α

Circle centre O with radius r

Angle of sector is 2α

G lies on the axis of symmetry and

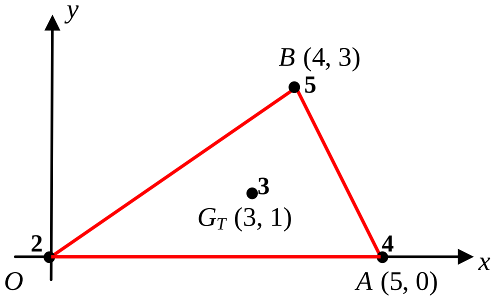
$$OG = \frac{2r\sin\alpha}{3\alpha}$$



Example 1: A uniform triangular lamina has mass 3 *kg*, and its vertices are *O* (0, 0), *A* (5, 0) and *B* (4, 3). Masses of 2, 4 and 5 *kg* are attached at *O*, *A* and *B* respectively.

Find the centre of mass of the system.

Solution: First find the centre of mass of the triangle, G_T.



$$\underline{\bar{r}}_T = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c}) = \frac{1}{3}\left(\binom{0}{0} + \binom{5}{0} + \binom{4}{3}\right)$$

$\Rightarrow$ G_T is (3, 1).

We can now think of 4 masses, 2 kg at *O*, 4 *kg* at *A*, 5 kg at *B* and 3 *kg* (the triangle) at G_T.

$$M\,\underline{\bar{r}} = \sum m_i\,\underline{r}_i \quad \Rightarrow \quad (2 + 4 + 5 + 3)\,\underline{\bar{r}} = 2\binom{0}{0} + 4\binom{5}{0} + 5\binom{4}{3} + 3\binom{3}{1}$$

$$\Rightarrow \quad 14\,\underline{\bar{r}} = \binom{49}{18}$$

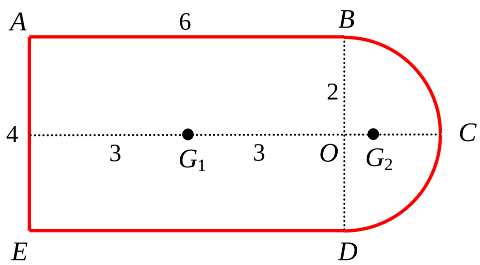
$$\Rightarrow \quad \text{G is at } \left(\frac{49}{14}, \frac{18}{14}\right).$$

Centres of mass of composite laminas

In the following examples ρ is the surface density, or mass per unit area.

Example 3: Find the centre of mass of a uniform rectangular lamina attached to a uniform semi circular lamina, as shown in the diagram.

O is the centre of the semi-circle.



Solution: The centre of mass will lie on the line of symmetry, *OC*, so we only need to find the horizontal distance of the centre of mass from *O*.

By symmetry the centre of mass of the rectangle is at G_1, as shown.

The semi-circle is a sector with angle $2 \times \frac{\pi}{2}$

$$\Rightarrow \quad OG_2 = \frac{2r\sin\alpha}{3\alpha} = \frac{2\times 2\sin\frac{\pi}{2}}{3\times\frac{\pi}{2}} = \frac{8}{3\pi}$$

Note $OG_1 = -3$ and $OG_2 = +\frac{8}{3\pi}$ negative and positive

	Rectangle	Semi-circle	Composite
mass	24ρ	$2\pi\rho$	$(24 + 2\pi)\rho$
distance of G from O	-3	$\frac{8}{3\pi}$	$\bar{x}$

$$\sum m_i x_i = M\bar{x} \quad \Rightarrow \quad 24\rho \times (-3) \quad + \quad 2\pi\rho \times \frac{8}{3\pi} \quad = \quad (24 + 2\pi)\rho \times \bar{x}$$

$$\Rightarrow \quad \bar{x} = \frac{-72+\frac{16}{3}}{24+2\pi} = \frac{-100}{3(12+\pi)} = -2{\cdot}201441691 = -2{\cdot}20 \quad \text{to 3 S.F.}$$

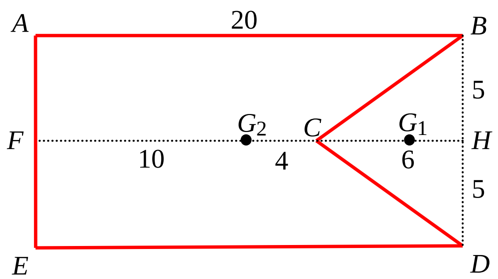
The centre of mass lies on the line of symmetry, 2·20 on the left of O (inside the rectangle).

Example 4:

Find the centre of mass of the uniform lamina *ABCDE*.

FC is an axis of symmetry. $CB = CD$.

$AE = 10$, $AB = 20$, $FC = 14$.



Solution: Think of the triangle *BCD* combining with this shape to form the rectangle *ABDE*.

Mass of rectangle = 200ρ, mass of triangle = $\frac{1}{2} \times 10 \times 6 = 30\rho \Rightarrow$ mass of shape = 170ρ

The centre of mass lies on the line of symmetry, *FH*.

Let G_2 be the centre of mass of the rectangle $\Rightarrow FG_2 = 10$

Let G_1 be the centre of mass of the triangle.

G_1 divides CH in the ratio 2 : 1 and $CH = 6$

$\Rightarrow \quad CG_1 = 4$ and $G_1H = 2$

$\Rightarrow \quad FG_1 = 10 + 4 + 4 = 18$

mass	170ρ	30ρ	200ρ
distance of G from AE	$FG = x$	$FG_1 = 18$	$FG_2 = 10$

$$\sum m_i x_i = M\bar{x} \quad \Rightarrow \quad 170\rho\, x \quad + \quad 30\rho \times 18 \quad = \quad 200\rho \times 10$$

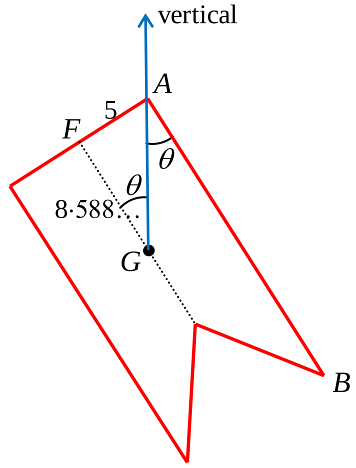
$$\Rightarrow \quad 170x = 2000 - 540 = 1460$$

$$\Rightarrow \quad x = \frac{1460}{170} = 8{\cdot}588235294 = 8{\cdot}59 \quad \text{to 3 S.F.}$$

The centre of mass lies on *FC* at a distance 8·59 from *AE*.

Laminas suspended freely under gravity

Example 5: The lamina in example 4 is suspended from A, and hangs in equilibrium. Find the angle made by AB with the downward vertical.



Solution: **G must be vertically below *A*.**

This must be stated in any solution (method).

From example 4 we know that

$FG = 8{\cdot}588\ldots$ and $AF = 5$

The angle made by AB with the downward vertical is

$$\theta = \arctan\left(\frac{5}{8.588\ldots}\right) = 30{\cdot}20761248$$

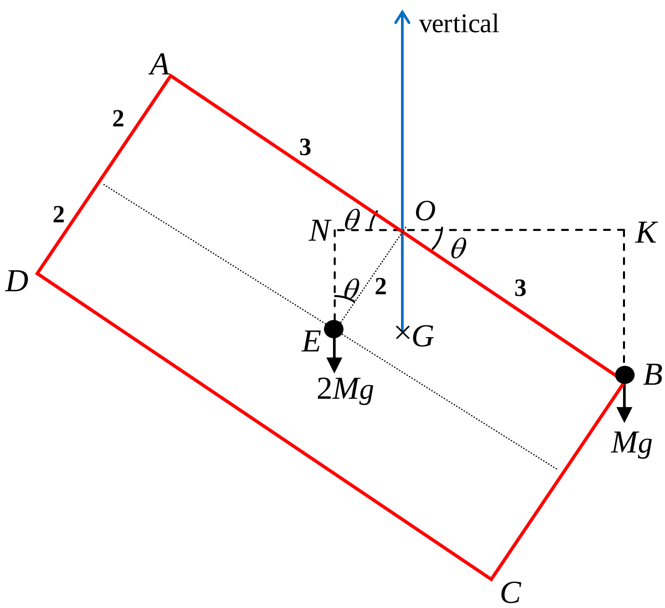
$\Rightarrow$ angle made by AB with the downward vertical is $30{\cdot}2^{\circ}$ to the nearest $0{\cdot}1^{\circ}$.

Body with point mass hanging freely

The best technique will probably be to take moments about the point of suspension.

Example: A uniform rectangular lamina $ABCD$, where $AB = 6$ m and $BC = 4$ m, has mass $2M$. A particle of mass M is attached to the edge of the rectangle at B.

The compound body is freely suspended under gravity from O, the mid-point of AB. Find the angle made by AB with the horizontal.



Solution: As usual a *good, large diagram* is essential.

We think of the rectangle as a point mass, $2M$, at E, centre of symmetry.

Let the angle made by AB with the horizontal be θ, then $\angle OEN = \theta$.

The perpendicular distance from O to the line of action of $2Mg$ is $ON = 2\sin\theta$, and

the perpendicular distance from O to the line of action of Mg is $OK = 3\cos\theta$

Taking moments about O

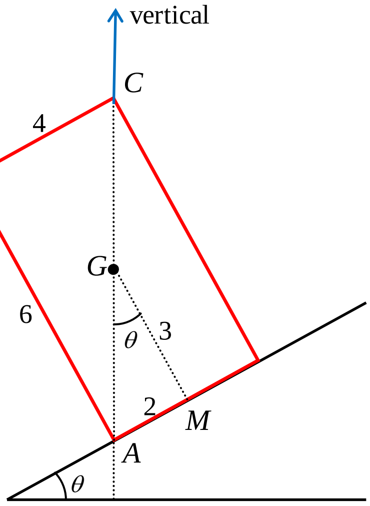
$$2Mg \times 2\sin\theta = Mg \times 3\cos\theta$$

$$\Rightarrow \quad \tan\theta = \frac{3}{4}$$

$$\Rightarrow \quad \theta = 36{\cdot}9^{\circ}.$$

Toppling on a slope

Example 6: What angle of slope would cause a 4 × 6 uniform rectangular lamina to topple (assuming that the friction is large enough to prevent sliding).



Solution:

G must be vertically above A when the lamina is on the point of toppling.

This must be stated in any solution (method).

By angle theory $\angle AGM = \theta$, the angle of the slope.

$\tan\theta = \frac{2}{3}$ $\Rightarrow$ $\theta = 33{\cdot}69006753$

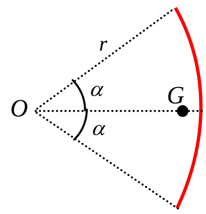
The lamina will topple when the angle of slope exceeds $33{\cdot}7^\circ$, to nearest $0{\cdot}1^\circ$.

Centres of mass of wire frameworks.

In the following examples ρ is mass per unit length.

1) The centre of mass of a uniform straight wire is at its centre.

2) The centre of mass of a uniform circular arc, of radius r and angle at the centre 2α, lies on the axis of symmetry and $OG = \dfrac{r\sin\alpha}{\alpha}$.

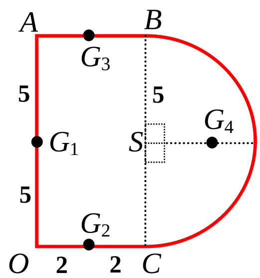


Example 7: A uniform wire **framework** is shown in the diagram.

$OABC$ is a rectangle and the arc is a semi-circle.

OA = 10 and OC = 4.

Find the position of the centre of mass.



Solution: With frameworks it is often easier to use vectors. Take the origin at O.

The semi-circlular arc has angle $2 \times \frac{\pi}{2}$

$\Rightarrow$ centre of mass is at G_4, where $SG_4 = \dfrac{r\sin\alpha}{\alpha} = \dfrac{5\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \dfrac{10}{\pi}$

$\Rightarrow$ semi-circular arc has mass $5\pi\rho$ at $\left(4 + \frac{10}{\pi}, 5\right)$.

We now consider each straight wire as a point mass at the midpoint of the wire.

	OA	OC	AB	semi-circle
mass	10ρ	4ρ	4ρ	$5\pi\rho$
centre of mass	$\begin{pmatrix}0\\5\end{pmatrix}$	$\begin{pmatrix}2\\0\end{pmatrix}$	$\begin{pmatrix}2\\5\end{pmatrix}$	$\begin{pmatrix}4+\dfrac{10}{\pi}\\5\end{pmatrix}$

$M\,\underline{\bar{\boldsymbol{r}}} = \sum m_i\,\underline{\boldsymbol{r}}_i \;\Rightarrow\quad (18+5\pi)\rho\,\underline{\bar{\boldsymbol{r}}} = 10\rho\begin{pmatrix}0\\5\end{pmatrix} + 4\rho\begin{pmatrix}2\\0\end{pmatrix} + 4\rho\begin{pmatrix}2\\5\end{pmatrix} + 5\pi\rho\begin{pmatrix}4+\frac{10}{\pi}\\5\end{pmatrix}$

$\Rightarrow\qquad \underline{\bar{\boldsymbol{r}}} = \frac{1}{18+5\pi}\begin{pmatrix}66+20\pi\\70+25\pi\end{pmatrix} = \begin{pmatrix}3\cdot 822000518\\5\end{pmatrix} = \begin{pmatrix}3\cdot 82\\5\end{pmatrix}$ to 3 S.F.

3 Work, energy and power

Definitions

$$v^2 = u^2 + 2as$$

$$\Rightarrow \quad mas = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2 \qquad \text{re-arranging and multiplying by } m$$

$$\Rightarrow \quad Fs = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$$

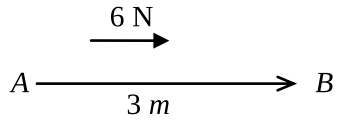
We define

- The kinetic energy, K.E., of a body of mass m moving with speed v is $\tfrac{1}{2}mv^2$.
- Work done by a (constant) force F
  is the magnitude of the force $\times$ distance moved in the direction of the force.
- The units of kinetic energy and work are Joules, J.

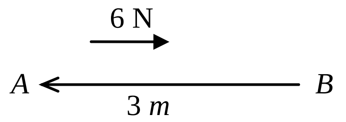
Work done by a (constant) force.

Forces parallel to the displacement

(*a*) Work done by a force of 6 N when a particle is moving **from *A* to *B*** (AB = 3 m) in the direction of the force,
is $6 \times (+3) = 18$ J



(*b*) Work done by a force of 6 N when a particle is moving **from *B* to *A*** in the opposite direction to the force,
is $6 \times (-3) = -18$ J



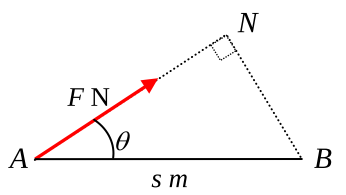
Forces at an angle to the displacement

Work done can be calculated in two ways

(*a*) As a body moves from A to B, we can think that it has moved through the distance AN in the direction of the force.

$AN = s\cos\theta$,

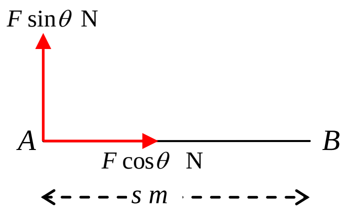
$\Rightarrow$ work done is $F \times s\cos\theta = Fs\cos\theta$.



(*b*) Or, we can resolve the force F parallel and perpendicular to the displacement.

The component $F\sin\theta$ is perpendicular to the displacement, and so does no work.

The component $F\cos\theta$ is parallel to the displacement,
$\Rightarrow$ work done is $F\cos\theta \times s = Fs\cos\theta$, as in part (*a*).



Work done by gravity

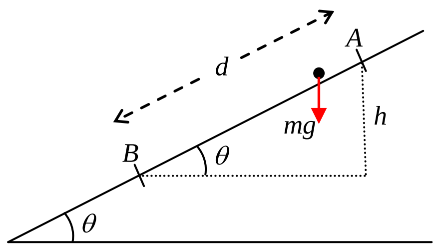
If a particle of mass m falls a vertical distance h, then the work done by gravity is mgh,

force and displacement are in the same direction

If a particle of mass m rises a vertical distance h, then the work done by gravity is $-mgh$,

force and displacement are in opposite directions

When a particle is moving on a slope, it is usual to consider the vertical distance moved and multiply by mg to calculate the work done by gravity.



From A to B the particle moves a distance d, but its vertical movement is $h = d\sin\theta$

$\Rightarrow$ work done by gravity $= mgh = mgd\sin\theta$

Work done by gravity is also known as loss or gain in Gravitational Potential Energy, G.P.E.

Work-energy equation

The equation $Fs = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$ can be re-arranged as

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + Fs$$

which can be thought of as

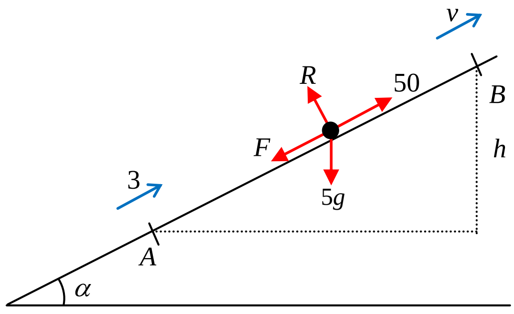
Final K.E. = Initial K.E. $\pm$ Work done

Notice the $\pm$

- If a force increases the K.E. then the work done is **positive**
- If a force decreases the K.E. then the work done is **negative**

Example 1: A particle of mass 5 kg is being pulled up a rough slope by a force of 50 N parallel to the slope. The coefficient of friction is 0·2, and the slope makes an angle of $\alpha = \tan^{-1}\left(\frac{3}{4}\right)$ with the horizontal.

The particle is observed to be moving up the slope with a speed of 3 ms^{-1}. Find its speed when it has moved 12 m up the slope.



Solution:

$AB = 12 \Rightarrow \quad h = 12 \sin\alpha = 6$

Resolve ↘ $\quad R = 5g\cos\alpha = 4g$

Moving $\Rightarrow \quad F = \mu R = 0{\cdot}2 \times 4g = 0{\cdot}8g$

Work done by R = 0 ($\perp$ to motion)

Work done by $F = -\,0{\cdot}8g \times 12 = -\,9.6g$ J — reduces K.E. so negative

Work done by 50 N = 50 × 12 = 600 J — increases K.E. so positive

Work done by gravity (G.P.E.) = $-mgh = -5g \times 6 = -30g$ — uphill, reduces K.E. so negative

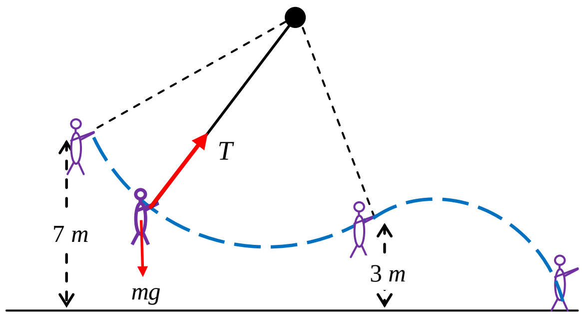
Work-energy equation Final K.E. = Initial K.E. ± Work done

$$\tfrac{1}{2} \times 5v^2 = \tfrac{1}{2} \times 5 \times 3^2 - 9{\cdot}6g + 600 - 30g$$

$\Rightarrow \quad v^2 = 93{\cdot}768 \quad \Rightarrow \quad v = 9{\cdot}683387837 = 9{\cdot}7$ to 2 S.F.

Particle is moving at 9·7 ms^{-1} after it has travelled 12 m.

Example 2: Tarzan swings on a rope, lets go and falls to the ground. If Tarzan was initially 7 m above the ground and not moving, and if he lets go when he is 3 m above the ground after the rope has passed the vertical, with what speed does he hit the ground?



Solution:

The only forces acting on Tarzan are the tension in the rope, T, and gravity, mg.

The work done by T is 0, since T is always perpendicular to the motion

Work done by gravity (G.P.E.) = mgh = $mg \times 7$ — downwards, increases K.E. so positive

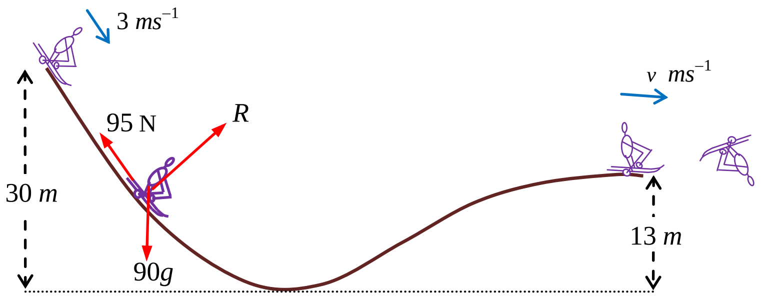
Work-energy equation Final K.E. = Initial K.E. ± Work done

$$\tfrac{1}{2} \times mv^2 = 0 + 7mg$$

$\Rightarrow \quad v^2 = 14g \quad \Rightarrow \quad v = 11.71324037 = 12\ ms^{-1}$ to 2 S.F.

Note that it does not matter when he lets go of the rope. This will only affect the direction in which he is moving when he hits the ground, **not** his speed.

Example 3: Tarzan now goes skiing in Switzerland. It is much colder than Africa so he is wearing lots of warm clothes and his mass is 90 *kg*. He starts with a speed of 3 ms^{-1} and skis along a path as shown in the diagram when he comes to a cliff. The total length of his path is 150 *m*, and he experiences a constant resistance of 95 N. Find his speed as he launches himself into thin air.



Solution:

Height lost between start and finish is $30 - 13 = 17$ metres

$\Rightarrow$ Work done by gravity (GPE) = $mgh = 90 \times g \times 17 = 1530g$ — increases K.E. so positive

Work done by $R = 0$ — R is always perpendicular to the motion

Work done by 95 N $= Fs = -95 \times 150 = -14250$ — decreases K.E. so negative

Work-energy equation Final K.E. = Initial K.E. $\pm$ Work done

$$\tfrac{1}{2} \times 90v^2 = \tfrac{1}{2} \times 90 \times 3^2 + 1530g - 14250$$

$\Rightarrow$ $v^2 = 25{\cdot}533\ldots$ $\Rightarrow$ $v = 5{\cdot}053051883 = 5{\cdot}1$ ms^{-1} to 2 S.F.

Potential energy

When a body falls it gains K.E. The higher its starting point the greater the gain in K.E. We say that the *Gravitational Potential Energy*, G.P.E., of a body depends on its height above some fixed point.

When a body falls a distance *h*, the loss in G.P.E. is the work done by gravity, *mgh*.

$\Rightarrow$ $\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + mgh$ — loss in G.P.E. increases K.E. so positive

$\Rightarrow$ $mgh = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2$ or Loss in G.P.E. = Gain in K.E.

Similarly when a body rises a distance *h*, the gain in G.P.E. = *mgh*

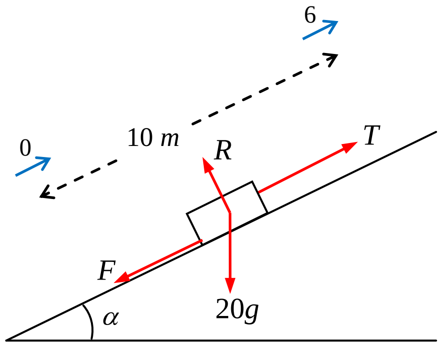
$\Rightarrow$ $\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 - mgh$ — gain in K.E. decreases K.E. so negative

$\Rightarrow$ $mgh = \tfrac{1}{2}mu^2 - \tfrac{1}{2}mv^2$ or Gain in G.P.E. = Loss in K.E.

You will do better using the energy equation than "gain in GPE = loss in KE", etc.

You are expected to understand the terms loss in G.P.E. and gain in G.P.E., but all you can always use work done by gravity if you wish.

Example 4: A block, of mass 20 *kg*, is pulled up a rough slope of angle α, $\tan^{-1}\left(\frac{3}{4}\right)$, by a rope parallel to the slope. The block starts from rest and is moving at 6 ms^{-1} when it has moved 10 *m*. If the coefficient of friction is $\mu = 0{\cdot}3$, find the tension in the rope.



Solution:

R ↖ $R = 20g\cos\alpha = 16g$

moving $\Rightarrow F = \mu R = 0{\cdot}3 \times 16g = 4{\cdot}8g$

Work done by $R = 0$ ($\perp$ to motion)

Work done by $F = Fs = 4{\cdot}8g \times (-10) = -48g$ decreases K.E. so negative

Gain in G.P.E. $= mgh = 20g \times (-10\sin\alpha) = -120g$ decreases K.E. so negative

Work done by tension $= Ts = 10T$ increases K.E. so positive

Work-energy equation Final K.E. = Initial K.E. ± Work done

$$\frac{1}{2} \times 20 \times 6^2 = 0 - 48g - 120g + 10T$$

$\Rightarrow$ $T = 200{\cdot}64 = 200$ N to 2 S.F.

Notice that we could have thought of work against by gravity, *mgh*, instead of gain in G.P.E.

Power

Power is the rate of doing work. The units are Watts = Joules/second.

For a *constant* force, F, moving a distance s the work done is $W = Fs$

$\Rightarrow$ the power, $P = \frac{d}{dt}(Fs) = F\frac{ds}{dt} = Fv$ since F is constant

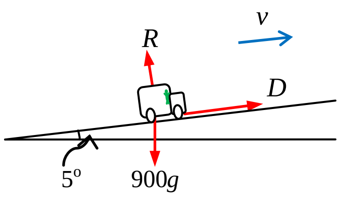
$\Rightarrow$ the power developed by a *constant* force F moving its point of application at a speed v is $P = Fv$.

Example 5: A car of mass 900 *kg* is travelling up a slope of 5° at a constant speed. Assume that there is no resistance to motion – other than gravity.

(a) If the engine of the car is working at a rate of 20 *kW*, find the speed of the car.

(b) The car later travels up a slope of 8° at the same constant speed. Find the power developed by the engine.

Solution:



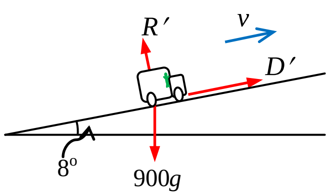
(a) Power, $P = 20\,000 = Dv$

$\Rightarrow$ $v = \frac{20\,000}{D}$

Constant speed $\Rightarrow$

$R \nearrow$ $\Rightarrow$ $D = 900g \sin 5^\circ$

$\Rightarrow$ $v = \frac{20\,000}{900g\sin 5^o} = 26{\cdot}01749035 = 26\ ms^{-1}$ to 2 S.F.



(b) $v = \frac{20\,000}{900g\sin 5^o}$ from part (a)

$R \nearrow$ $\Rightarrow$ $D' = 900g \sin 8^\circ$

$\Rightarrow$ power developed $= D'v$

$\Rightarrow$ $P = 900g \sin 8^\circ \times \frac{20\,000}{900g\sin 5^o}$

$\Rightarrow$ $P = 20\,000 \times \frac{\sin 8^o}{\sin 5^o} = 31936{\cdot}64504$

$\Rightarrow$ power developed by the engine is 32 kW to 2 S.F.

4 Collisions

Impulse and momentum – using vectors

Impulse = change in momentum

$$\underline{\boldsymbol{I}} = m\,\underline{\boldsymbol{v}} - m\,\underline{\boldsymbol{u}}$$

Conservation of linear momentum

If there is no external impulse during a collision, then the total momentum before impact equals the total momentum after impact.

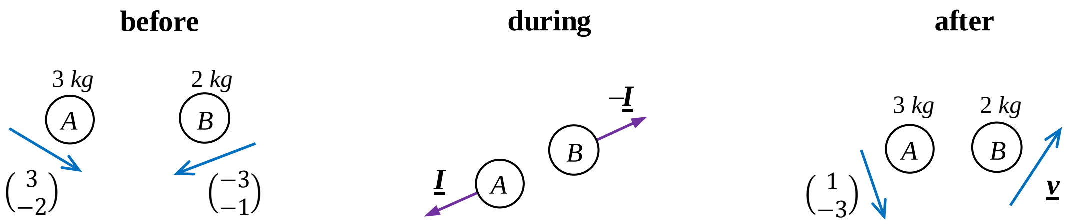
$$m_1\,\underline{\boldsymbol{u}}_1 + m_2\,\underline{\boldsymbol{u}}_2 = m_1\,\underline{\boldsymbol{v}}_1 + m_2\,\underline{\boldsymbol{v}}_2$$

Example 1: A ball, *A*, of mass 3 kg is moving with velocity $\begin{pmatrix}3\\-2\end{pmatrix}$ ms^{-1} when it collides with another ball, *B*, of mass 2 kg moving with velocity $\begin{pmatrix}-3\\-1\end{pmatrix}$ ms^{-1}. After the collision *A* moves with velocity $\begin{pmatrix}1\\-3\end{pmatrix}$ ms^{-1}.

Find the velocity of *B* after the collision, and the impulse **on *A*** during the collision.

As usual it is **essential** to draw **good diagrams** and to take care over positive and negative values.

Solution: There is no external impulse, so momentum is conserved



CLM

$$3 \times \begin{pmatrix}3\\-2\end{pmatrix} + 2 \times \begin{pmatrix}-3\\-1\end{pmatrix} = 3 \times \begin{pmatrix}1\\-3\end{pmatrix} + 2 \times \underline{\boldsymbol{v}}$$

$$\Rightarrow \quad \underline{\boldsymbol{v}} = \begin{pmatrix}0\\0\cdot5\end{pmatrix}\ ms^{-1}$$

For *A* $\underline{\boldsymbol{I}} = m\,\underline{\boldsymbol{v}} - m\,\underline{\boldsymbol{u}}$

$$\Rightarrow \quad \underline{\boldsymbol{I}} = 3 \times \begin{pmatrix}1\\-3\end{pmatrix} - 3 \times \begin{pmatrix}3\\-2\end{pmatrix} = \begin{pmatrix}-6\\-3\end{pmatrix}$$

$\Rightarrow$ impulse on *A* is $\begin{pmatrix}-6\\-3\end{pmatrix}$ *Ns*

Note: the impulse on *B* is $-\underline{\boldsymbol{I}} = \begin{pmatrix}+6\\+3\end{pmatrix}$ *Ns*

Newton's law of restitution

This is also known as **Newton's Experimental Law, NEL**

Coefficient of restitution

The coefficient of restitution in a collision, e, is defined as

$$e = \frac{\text{speed of separation}}{\text{speed of approach}} \qquad 0 \le e \le 1$$

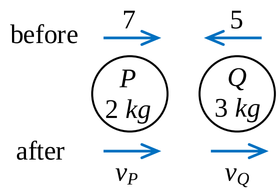
If $e = 1$ the collision is *perfectly elastic* and K.E. is conserved during the collision.

As usual it is **essential** to draw **good diagrams** and to take care over positive and negative values.

Example 2: Particles P and Q with masses 2 kg and 3 kg are moving towards each other with velocities of 7 ms^{-1} and 5 ms^{-1} respectively. If the coefficient of restitution is $\frac{3}{4}$, find the velocities of P and Q after the collision.

Solution: Let velocities of P and Q after the collision be v_P and v_Q.

We do not know their directions, so take them as marked in the diagram.



No external impulse $\Rightarrow$ CLM

$\rightarrow +$ $\quad\Rightarrow\quad 2 \times 7 + 3 \times (-5) = -1 = 2v_P + 3v_Q$ **I**

NEL $\quad\Rightarrow\quad e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$

$\Rightarrow \dfrac{3}{4} = \dfrac{v_Q - v_P}{7 - -5} \quad\Rightarrow\quad 9 = v_Q - v_P$ **II**

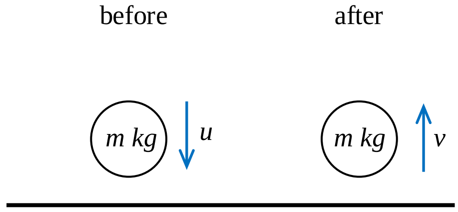
I $+ 2 \times$ **II** $\quad\Rightarrow\quad 17 = 5v_Q \Rightarrow v_Q = 3{\cdot}4$ and $v_P = -5{\cdot}6$.

$\Rightarrow$ P moves with speed 5·6 ms^{-1} in the **same** direction as its original direction

and Q moves with speed 3·4 ms^{-1} in the **opposite** direction to its original direction

Collisions with a plane surface

The velocity of the surface before and after is 0



NEL, $e = \dfrac{\text{speed of separation}}{\text{speed of approach}} = \dfrac{v - 0}{u - 0} = \dfrac{v}{u}$

Multiple collisions

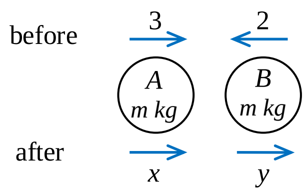
Treat each collision as a new problem – the final velocities from one collision become the initial velocities for the next collision.

Example 3: Two particles, A and B, are of equal mass and are moving towards each other with speed of 3 ms^{-1} and 2 ms^{-1} respectively and collide. Particle B then strikes a plane surface which is perpendicular to it direction of motion and rebounds. The coefficient or restitution between the two particles is $\frac{1}{2}$, and between B and the plane surface is $\frac{5}{7}$.

Show that B collides with A a second time, and find the velocities of both particles after this collision.

Solution: Let masses of particles be m kg.

First collision, A and B $\rightarrow +$



CLM $\Rightarrow$ $3m - 2m = mx + my$

$\Rightarrow$ $1 = x + y$ **I**

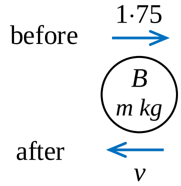
NEL $\Rightarrow$ $e = \frac{1}{2} = \frac{y-x}{3--2}$

$\Rightarrow$ $2{\cdot}5 = y - x$ **II**

I + II $\Rightarrow$ $y = 1{\cdot}75$ and $x = -0{\cdot}75$

A moves at $0{\cdot}75$ ms^{-1} $\leftarrow$, and B moves at $1{\cdot}75$ ms^{-1} $\rightarrow$.

Second collision, B with plane surface



NEL $\Rightarrow$ $e = \frac{5}{7} = \frac{v-0}{1{\cdot}75-0}$

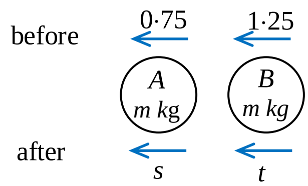
$\Rightarrow$ $v = 1{\cdot}25$

After this second collision, B moves at $1{\cdot}25$ ms^{-1} $\leftarrow$, and A is still moving at $0{\cdot}75$ ms^{-1} $\leftarrow$.

B is moving faster than A in the same direction $\Rightarrow$ there will be a third collision.

Third collision, A and B $+ \leftarrow$

CLM $\Rightarrow$ $0{\cdot}75m + 1{\cdot}25m = ms + mt$



$\Rightarrow$ $2 = s + t$ **I**

NEL $\Rightarrow$ $e = \frac{1}{2} = \frac{s-t}{1{\cdot}25-0{\cdot}75}$

$\Rightarrow$ $\frac{1}{4} = s - t$ **II**

I + II $\Rightarrow$ $s = \frac{9}{8}$ and $t = \frac{7}{8}$

A moves at $\frac{9}{8}$ ms^{-1} $\leftarrow$, and B moves at $\frac{7}{8}$ ms^{-1} $\leftarrow$, both moving away from the plane surface.

Kinetic energy and impulses/collisions

K.E. will be generated if there is an external impulse, but in collisions K.E. will be lost (unless the collision is perfectly elastic, $e = 1$).

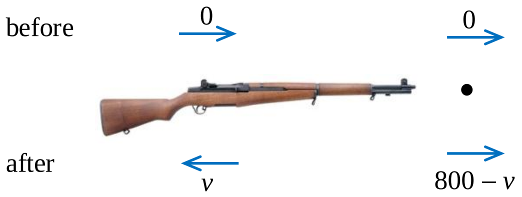
Example 4: A rifle of mass 5 *kg* fires a bullet of mass 25 *grams* with a muzzle velocity of 800 ms^{-1}. The rifle is pointing in a horizontal direction and is free to move.

Find the K.E. generated in firing the rifle.

Solution: Linear momentum will be conserved and the rifle will move in the opposite direction to the bullet. Note that the muzzle velocity of the bullet is the velocity *relative to the rifle.*

Let the velocity of the rifle be v ms^{-1}, then the **actual** velocity of the bullet will be $(800 - v)$ ms^{-1}.

$\rightarrow +$



CLM $\Rightarrow$ $0 = 0{\cdot}025 \times (800 - v) - 5v$

$\Rightarrow$ $v = \frac{20}{5{\cdot}025} = 3{\cdot}980099502$

velocity of bullet = 796·0199005

K.E. before = 0

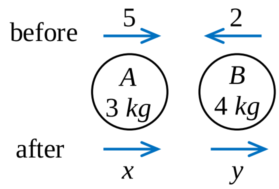
K.E. after $= \frac{1}{2} \times 5 \times (3{\cdot}98\ldots)^2 + \frac{1}{2} \times 0{\cdot}025 \times (796{\cdot}\ldots)^2 = 7960{\cdot}199005$

K.E. generated is 7960 J to 3 S.F.

Example 5: Particle A, mass 3 *kg*, and particle B, mass 4 *kg*, are moving towards each other with speeds of 5 ms^{-1} and 2 ms^{-1} respectively. If $e = \frac{1}{2}$, find the K.E. lost in the collision.

Solution:

There is no external impulse $\rightarrow +$



CLM $\Rightarrow$ $3 \times 5 - 4 \times 2 = 3x + 4y$

$\Rightarrow$ $7 = 3x + 4y$ **I**

NEL $\Rightarrow$ $e = \frac{1}{2} = \frac{y - x}{5 + 2}$

$\Rightarrow$ $7 = 2y - 2x$ **II**

I $- 2 \times$ **II** $\Rightarrow$ $-7 = 7x$

$\Rightarrow$ $x = -1$ and $y = 2{\cdot}5$

K.E. lost $= \left[\frac{1}{2} \times 3 \times 5^2 + \frac{1}{2} \times 4 \times 2^2\right] - \left[\frac{1}{2} \times 3 \times 1^2 + \frac{1}{2} \times 4 \times 2{\cdot}5^2\right]$

$\Rightarrow$ K.E. lost $= 45{\cdot}5 - 14 = 31{\cdot}5$ J

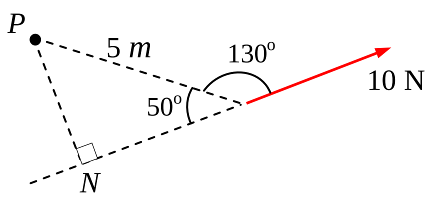
5 Statics of rigid bodies

Moment of a force

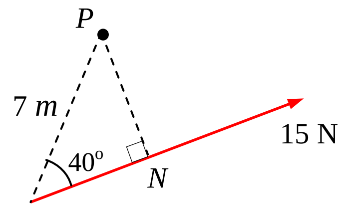
The *moment* of a force *F* about a point *P* is the magnitude of the force multiplied by the perpendicular distance from *P* to the line of action of *F*.

Examples:

(i)



(ii)



Moment $= 10 \times PN$

$= 10 \times 5 \sin 50^\circ$

$= 50 \sin 50^\circ$ *Nm*

Moment $= 15 \times PN$

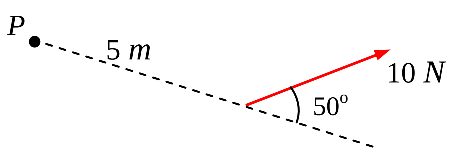
$= 15 \times 7 \sin 40^\circ$

$= 105 \sin 40^\circ$ *Nm*

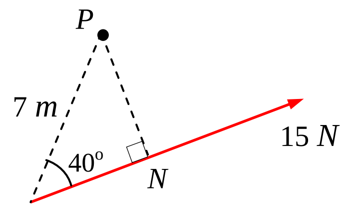
Alternative method

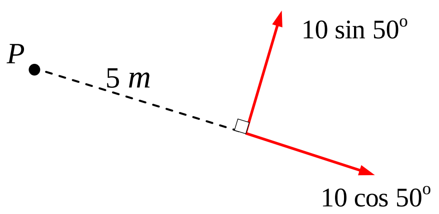
Resolve the force, *F*, in two directions – one component passing through *P*, and the other perpendicular to this.

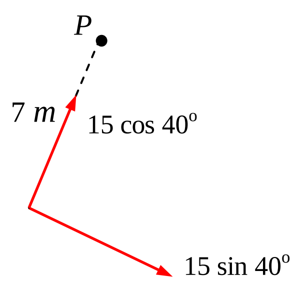
(i)



(ii)







Moment $= 10 \sin 50^\circ \times 5$

$= 50 \sin 50^\circ$ *Nm*

as before

Moment $= 15 \sin 40^\circ \times 7$

$= 105 \sin 40^\circ$ *Nm*

as before

You should remember **both** of these techniques (not neither!).

Equilibrium

A system of forces will be in equilibrium if

(i) The sum of the resolved forces in any direction is zero.

(ii) The moment about any point is zero.

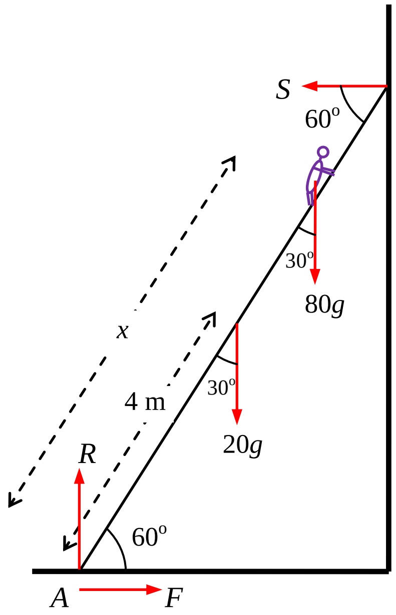
Example 1: A uniform ladder of mass 20 *kg* and length 8 *m* is leaning against a smooth vertical wall on rough ground. The ladder makes an angle of 60° with the ground, and the coefficient of friction between the ladder and the ground is 0·5.

What is the maximum height that a man of mass 80 *kg* can climb up the ladder before it starts to slip?

Solution:

The wall is smooth so the reaction, S, will be perpendicular to the wall.

At the man's highest point, x metres up the ladder, the friction will be at its maximum $\Rightarrow F = \mu R$



R $\uparrow$ $\Rightarrow$ $R = 80g + 20g = 100g$

R $\rightarrow$ $\Rightarrow$ $F = S$

Moments about A

$$20g \sin 30 \times 4 + 80g \sin 30 \times x = S \sin 60 \times 8$$

$$\Rightarrow \quad 40g + 40gx = 4\sqrt{3}\, S$$

But $S = F = \mu R = 0{\cdot}5\, R = 50g$

$$\Rightarrow \quad 40g + 40gx = 4\sqrt{3} \times 50g$$

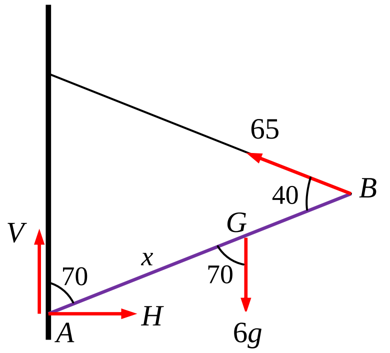
$$\Rightarrow \quad x = 5\sqrt{3} - 1 = 7{\cdot}660254038$$

The man can climb a distance of 7.7 *m* (to 2 S.F.) up the ladder before it starts to slip.

Example 2: A non-uniform rod AB, of length 4 m, is freely hinged to a vertical wall at A. It is held in equilibrium by a string which makes an angle of 40° with the rod, and is attached to the wall above A.

The tension in the string is 65 N, the mass of the rod is 6 kg and the rod makes an angle of 70° with the upwards vertical.

Find the position of the centre of mass of the rod, and the magnitude and direction of the reaction at the hinge.



Solution: Let the centre of mass of the rod, G, be x m from A, and let the reaction at the hinge have components V and H as shown (do not waste time worrying about the directions of V and H – they will be negative if you get the wrong direction).

$AB = 4\ m.$

Moments about A

$$6g \sin 70 \times x = 65 \sin 40 \times 4$$

$$\Rightarrow \quad x = 3{\cdot}024667934$$

$\Rightarrow$ the centre of mass is 3·0 m from A to 2 S.F.

R $\uparrow$ $\quad V + 65 \sin 20 = 6g \Rightarrow \quad V = 36{\cdot}56869068$ string is at 20° to horizontal

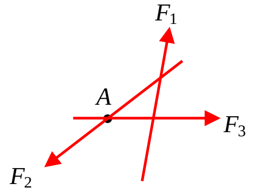
R $\rightarrow$ $\quad H = 65 \cos 20 = 61.08002035$

$$\Rightarrow \quad \text{magnitude} = \sqrt{V^2 + H^2} = 71{\cdot}19015399\ N$$

and angle above the horizontal $= \arctan\left(\frac{V}{H}\right) = 30{\cdot}90901406^{\circ}$

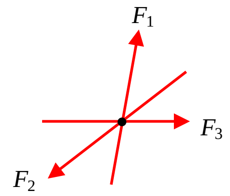
The reaction at the hinge is 71 N at an angle of 31° above the horizontal, 2 S.F.

Three non-parallel forces in equilibrium



If three forces are **not concurrent**, as shown in the diagram, then the moment about A, intersection of F_1 and F_2, can never be zero, and the forces **cannot** be in **equilibrium**.

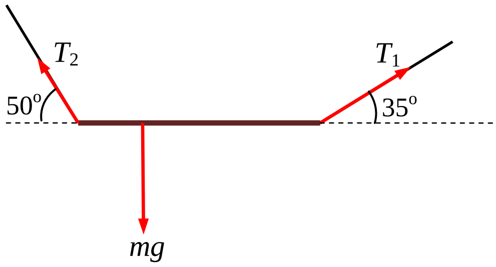
Thus, if three forces are in equilibrium, their lines of action must pass through one point.



Note that for the three forces to be in equilibrium, the sum of the resolved forces in any direction must also be zero.

Example 3: A non-uniform rod of length 6 *m* and mass *m kg* is supported at its ends by two strings, which make angles of 50° and 35° with the horizontal, as shown.

If the rod is horizontal and in equilibrium, find the position of its centre of mass.



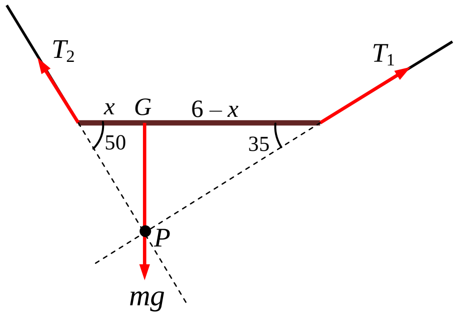
Solution: The three forces are in equilibrium, and therefore their lines of action must be concurrent.

If the directions of T_1 and T_2 meet at P, then mg must pass through P.

Now it is just trigonometry

$$GP = x \tan 50^\circ, \text{ and } GP = (6 - x) \tan 35^\circ$$

$$\Rightarrow \quad x = \frac{6 \tan 35}{\tan 50 + \tan 35} = 2{\cdot}220576925$$



Centre of mass is $2{\cdot}2$ *m*, 2 S.F., from end with the 50° angle.

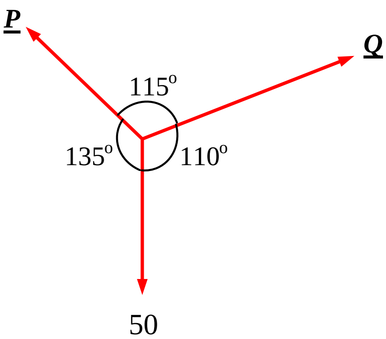
Triangle of forces

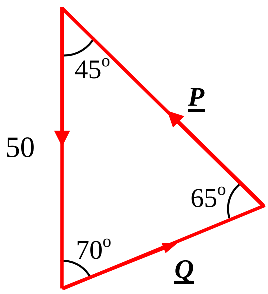
If three forces ***P***, ***Q*** and ***R*** are in equilibrium, then their (vector) sum must be zero.

Thus the three forces must form a triangle.

Example 4: The three forces shown are in equilibrium. Find the magnitudes of ***P*** and ***Q***.

Solution:





From the diagram we can draw a triangle of forces – check that the arrows go round the triangle in the same direction.

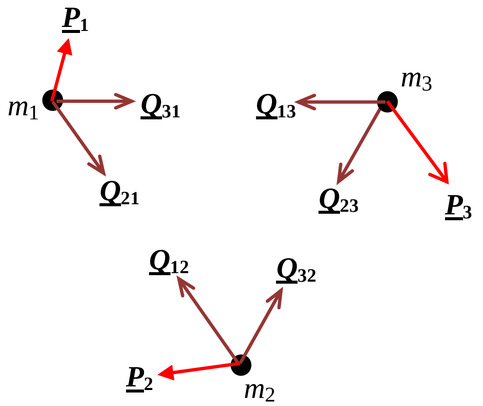
Sine Rule $\quad \Rightarrow \quad \dfrac{50}{\sin 65} = \dfrac{Q}{\sin 45} = \dfrac{P}{\sin 70}$

$$\Rightarrow \quad P = \frac{50 \sin 70}{\sin 65} = 51{\cdot}84180442 \text{ and } \quad Q = \frac{50 \sin 45}{\sin 65} = 39{\cdot}01030044$$

$$\Rightarrow \quad P = 52 \text{ N} \quad \text{and} \quad Q = 39 \text{ N} \quad \text{to 2 S.F.}$$

Appendix

Centre of mass of *n* particles



Consider three particles with masses m_1, m_2 and m_3 at points with position vectors $\underline{\mathbf{r}}_1$, $\underline{\mathbf{r}}_2$ and $\underline{\mathbf{r}}_3$.

Let the force of m_1 on m_2 be $\underline{\mathbf{Q}}_{12}$, of m_2 on m_1 be $\underline{\mathbf{Q}}_{21}$. Then $\underline{\mathbf{Q}}_{12}$ and $\underline{\mathbf{Q}}_{21}$ are internal forces and

$\underline{\mathbf{Q}}_{12} + \underline{\mathbf{Q}}_{21} = \underline{\mathbf{0}}$.

The other internal forces are defined in a similar way.

Let $\underline{\mathbf{P}}_1$, $\underline{\mathbf{P}}_2$ and $\underline{\mathbf{P}}_3$ be external forces on m_1, m_2 and m_3.

Newton's second law for each particle gives

$$\underline{\mathbf{P}}_1 + \underline{\mathbf{Q}}_{21} + \underline{\mathbf{Q}}_{31} = m_1\,\ddot{\underline{\mathbf{r}}}_1$$

$$\underline{\mathbf{P}}_2 + \underline{\mathbf{Q}}_{32} + \underline{\mathbf{Q}}_{12} = m_2\,\ddot{\underline{\mathbf{r}}}_2$$

$$\underline{\mathbf{P}}_3 + \underline{\mathbf{Q}}_{13} + \underline{\mathbf{Q}}_{23} = m_3\,\ddot{\underline{\mathbf{r}}}_3$$

Adding $\quad \underline{\mathbf{P}}_1 + \underline{\mathbf{P}}_2 + \underline{\mathbf{P}}_3 = m_1\,\ddot{\underline{\mathbf{r}}}_1 + m_2\,\ddot{\underline{\mathbf{r}}}_2 + m_3\,\ddot{\underline{\mathbf{r}}}_3 \qquad$ **I** $\qquad$ all internal forces, Q, cancel

Define $\quad \underline{\mathbf{r}}_g = \dfrac{m_1\underline{\mathbf{r}}_1 + m_2\underline{\mathbf{r}}_2 + m_3\underline{\mathbf{r}}_3}{m_1+m_2+m_3} \quad \Rightarrow \quad \ddot{\underline{\mathbf{r}}}_g = \dfrac{m_1\ddot{\underline{\mathbf{r}}}_1 + m_2\ddot{\underline{\mathbf{r}}}_2 + m_3\ddot{\underline{\mathbf{r}}}_3}{m_1+m_2+m_3}$

$$m_1\ddot{\underline{\mathbf{r}}}_1 + m_2\ddot{\underline{\mathbf{r}}}_2 + m_3\ddot{\underline{\mathbf{r}}}_3 = (m_1+m_2+m_3)\,\underline{\mathbf{r}}_g \qquad \textbf{II}$$

From **I** and **II** $\quad \underline{\mathbf{P}}_1 + \underline{\mathbf{P}}_2 + \underline{\mathbf{P}}_3 = (m_1+m_2+m_3)\,\underline{\mathbf{r}}_g$

$\Rightarrow$ the point $\underline{\mathbf{r}}_g$ moves as if **all** the mass was concentrated at that point, and **all** the external forces acted at that point. This point, $\underline{\mathbf{r}}_g$, is called the *centre of mass.*

This can be generalised for *n* particles to give

$$M\,\underline{\mathbf{r}}_g = \sum m_i\,\underline{\mathbf{r}}_i$$

$$\Leftrightarrow \quad M\begin{pmatrix}\bar{x}\\ \bar{y}\end{pmatrix} = \sum m_i \begin{pmatrix}x_i\\ y_i\end{pmatrix}$$

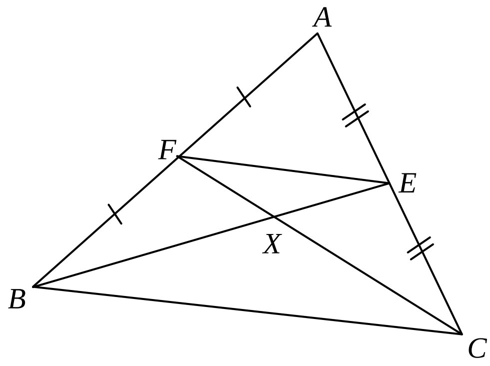
$$\Leftrightarrow \quad M\,\bar{x} = \sum m_i\,x_i \quad \text{and} \quad M\,\bar{y} = \sum m_i\,y_i$$

where $\boldsymbol{M}$ is the total mass, $\boldsymbol{M} = \sum m_i$

Medians of a triangle

A median of a triangle is a line joining one vertex to the mid-point of the opposite side.

Let BE and CF be medians of the triangle ABC.



F and E are the mid-points of the sides AC and AB

$\Rightarrow$ ΔABC is an enlargement of ΔAFE, scale factor 2

$\Rightarrow$ $FE = \frac{1}{2} BC$ and

FE is parallel to BC.

$\Rightarrow$ $\angle XFE = \angle XCB$ and $\angle XEF = \angle XBC$

$\Rightarrow$ triangles XFE and XCB are similar in the ratio 2 : 1

FX and XC are corresponding sides $\Rightarrow$ $FX = \frac{1}{2} XC$

also $EX = \frac{1}{2} XB$.

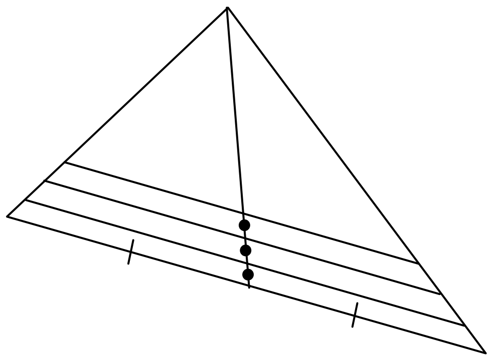
Thus X lies on the two medians, dividing each one in the ratio 2 : 1.

Similarly, X lies on the third median, AD, in the ratio 2 : 1

$\Rightarrow$ the medians meet in a point, X, which divides each median in the ratio 2 : 1.

Centre of mass of a triangle

Divide the triangle into narrow strips parallel to one side.



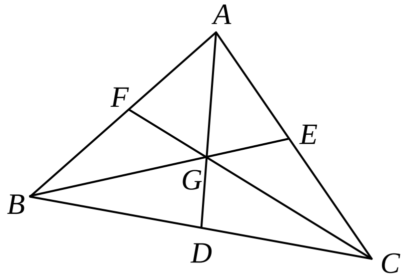
The centre of mass of each strip will be at the centre of each strip

$\Rightarrow$ the centre of mass of the triangle must lie on the line joining these centres of mass

$\Leftrightarrow$ the centre of mass lies on the median of the triangle.

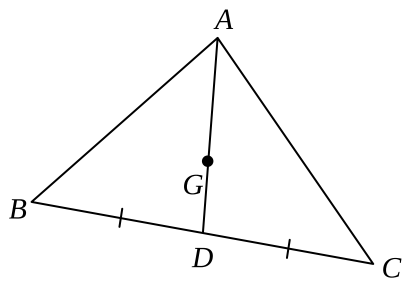
Similarly the centre of mass of the triangle lies on the other two medians, and therefore lies at the intersection of the medians.

$\Rightarrow$ G is the centre of mass of the triangle, and



$$AG : GD = BG : GE = CG : GF = 2 : 1$$

Centre of mass $\underline{g} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$



We know that the centre of mass of the triangle ABC lies on the median AD in the ratio $AG : GD = 2 : 1$.

A, B, C, D and G have position vectors $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ and $\underline{g}$.

D is the mid-point of BC

$\Rightarrow \quad \underline{d} = \frac{1}{2}(\underline{b} + \underline{c})$

$\Rightarrow \quad \overrightarrow{AD} = \underline{d} - \underline{a} = \frac{1}{2}(\underline{b} + \underline{c}) - \underline{a}$

$\quad \overrightarrow{AG} = \frac{2}{3}\overrightarrow{AD} = \frac{2}{3}\left(\frac{1}{2}(\underline{b} + \underline{c}) - \underline{a}\right) = \frac{1}{3}(\underline{b} + \underline{c}) - \frac{2}{3}\underline{a}$

$\Rightarrow \quad \underline{g} = \overrightarrow{OG} = \overrightarrow{OA} + \overrightarrow{AG} = \underline{a} + \frac{1}{3}(\underline{b} + \underline{c}) - \frac{2}{3}\underline{a}$

$\Rightarrow \quad \underline{g} = \frac{1}{3}(\underline{a} + \underline{b} + \underline{c})$

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