REVISION SHEET – MECHANICS 2

CENTRE OF MASS (CoM)

The main ideas are

	AQA	Edx	MEI	OCR
Uniform bodies	M2	M2	M2	M2
Composite bodies	M2	M2	M2	M2
CoM when suspended	M2	M2	M2	-

Before the exam you should know:

- How to find the centre of mass of a system of particles of given position and mass.
- How to locate centre of mass by appeal to symmetry.
- How to find the centre of a mass of a composite body by considering each constituent part as a particle at its centre of mass.
- How to use the position of the centre of mass in problems involving the equilibrium of a rigid body.

Centre of Mass

In much of the work that you have done involving moments, the whole weight of a rigid body, such as a rod, was considered to act at its centre. This is called the centre of mass. However, if the rod did not have uniform distribution of its mass, then the centre of mass would be at a different location.

Uniform Bodies

For a uniform body:

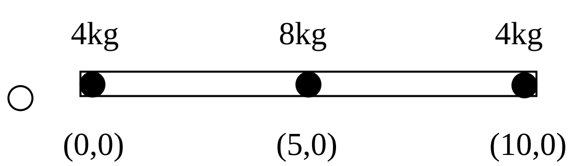
Moment of the whole mass at the centre of mass = sum of the moments of the individual masses

i.e. $(\sum m)\bar{x} = \sum(mx)$

A standard application of this method can be seen in the example to the right.

In examples using CoM it is always useful to check whether there is any symmetry present. (Although note that you should be careful to check that symmetry is actually present rather than assuming there is!)

For instance, in the example below, which is similar to the question to the right, it is possible to say that, by symmetry, the CoM will be at (5, 0).

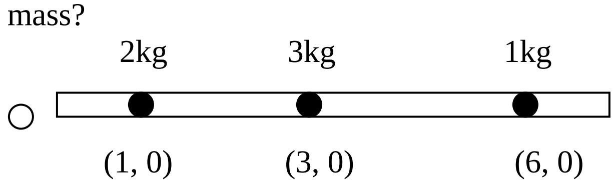


Example *(CoM)*
A rod A, of length 3 metres, has uniform mass. A rod B, also of length 3 metres, does not have uniform mass, with its mass concentrated towards one of its ends. For each rod, say whether it is possible to give the position of their CoM using the information give, If it is possible, state where the CoM is.

It can be clearly seen that for rod A the CoM acts half way along the rod, i.e. 1.5 m. However, for rod B, without further information you are unable to say where its CoM acts, except that because it is not uniformly distributed it will not lie in the centre.

Example *(Uniform bodies)*

Particles of mass 2kg, 3kg and 1kg are at the points (1, 0), (3, 0) and (6, 0) on a light rod which lies along the *x*-axis. What are coordinates of the centre of mass?



Relative to O:

$$(\sum m)\bar{x} = \sum(mx)$$

$$(2+3+1)\,\bar{x} = (2\times1)+(3\times3)+(1\times6)$$

$$6\bar{x} = 15$$

$$\bar{x} = 15/6 = 5/2$$

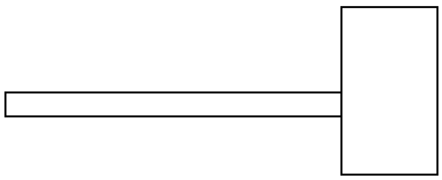
$$= 2.5$$

Therefore, the centre of mass lies at (2.5, 0).

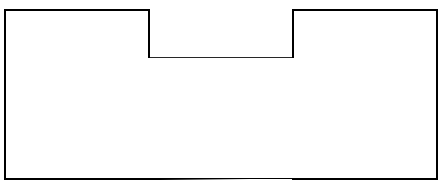
Composite Bodies

In the case for composite bodies you need to consider how 'individual bodies' can be put together to form the 'whole body'.

For instance a hammer may be represented by combining two bodies and considering the CoM of each of them:



A lamina like the one below could be considered as a number of separate bodies in several ways, i.e. one large rectangle (the whole length) plus the two smaller ones above, or three rectangles created by separating the three 'vertical' parts.



CoM when suspended *(not required for OCR)*

Within questions on CoM, once the actual CoM has been calculated then you may be asked to consider the body suspended from a specific point.

The key to these questions often lies in having a good, large clear diagram as in order to find the answer trigonometry is needed.

In the example to the right the lamina was turned so that when it was suspended from A, the line between A and the CoM was vertical. When doing this by hand it is better NOT to try and draw a whole new diagram, but simply to draw the line between the CoM and the relevant point. Once this line has been drawn careful consideration of the lengths and angles needs to be given, so that the required angle can be found.

Example *(Composite bodies)*

Find the coordinates of the centre of mass of five particles of mass 5kg, 4kg, 3kg, 2kg and 1kg situated at (4, 4), (7, 7) (1, 1), (7, 1) and (1, 7) respectively.

In order to solve this consider the *x* and *y* coordinates separately.

For the *x* coordinate:

$$(\sum m)\bar{x} = \sum(mx)$$

$$(5+4+3+2+1)\,\bar{x} = (5\times4)+(4\times7)+(3\times1)+(2\times7)+(1\times1)$$

$$15\bar{x} = 66$$

$$\bar{x} = {}^{66}\!/_{15} = 4.4$$

For the *y* coordinate:

$$(\sum m)\bar{y} = \sum(my)$$

$$(5+4+3+2+1)\,\bar{x} = (5\times4)+(4\times7)+(3\times1)+(2\times1)+(1\times7)$$

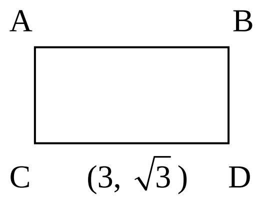
$$15\bar{y} = 60$$

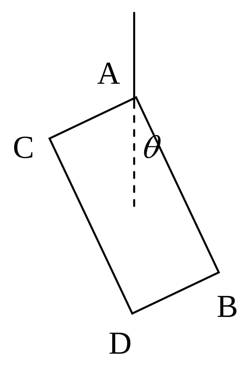
$$\bar{y} = {}^{60}\!/_{15} = 4$$

Therefore, the centre of mass lies at (4.4, 4).

Examples *(CoM when suspended)*

A regular lamina ABCD, whose CoM is at the point $(3, \sqrt{3})$, is suspended from point A. What angle does AB make with the vertical?





By trigonometry,

$$\tan\theta = \frac{\sqrt{3}}{3}$$

$$\theta = \tan^{-1}\frac{\sqrt{3}}{3} \;\; = \frac{\pi}{6}(\approx 0.52)$$

REVISION SHEET – MECHANICS 2

ENERGY, WORK & POWER

The main ideas are

	AQA	Edx	MEI	OCR
Work done	M2	M2	M2	M2
Kinetic energy	M2	M2	M2	M2
Gravitational potential energy	M2	M2	M2	M2
Power	M2	M2	M2	M2

Before the exam you should know:

- How to calculate the work done by a force.
- How to calculate kinetic energy and gravitational potential energy.
- The term mechanical energy and the work-energy principle.
- Be able to solve problems using the principle of conservation of energy.
- How to apply the concept of power to the solution of problems.

Work done

The work done by a constant force = force x distance moved in the direction of the force.
i.e. work done = Fs, where s is the distance. (Note: Sometimes you may see d used instead of s.)

However, if the displacement is not parallel to the force, then the force will need to be resolved to find the parallel component. i.e. if the force F acts at an angle θ to the parallel the Work done will equal $Fd\cos\theta$.

Kinetic energy

The energy possessed by a body because of its speed:

$$\text{Kinetic energy} = \frac{1}{2} \times \text{mass} \times (\text{speed})^2$$
$$= \frac{1}{2}mv^2$$

Also, in some questions it can be useful to use the idea that:
work done by a force = final KE – initial KE

Gravitational potential energy (GPE)

GPE is just one form of potential energy. You can think of potential energy as energy stored in an object, which gives it the potential to move when released. If a ball is dropped, the gravitational potential energy of the ball is converted into kinetic energy. As the height of the object decreases, the gravitational potential of the ball decreases, and its kinetic energy increases.

Example *(Work done)*
A bus on level ground is subject to a resisting force (from its brakes) of 10 kN for a distance of 400 m. How much energy does the bus lose?

The forward force is –10 000N.
The work done by it is –10 000 x 400
= –4 000 000 J
Hence –4 000 000 J of energy is lost and the bus slows down.

Example *(Kinetic energy)*
A bullet, of mass 30 grams, is fired at a wooden barrier 2.5 cm thick. When it hits the barrier it is travelling at 200 ms^{-1}. The barrier exerts a constant resistive force of 4500 N on the bullet. Does the bullet pass through the barrier and if so with what speed does it emerge?

$$\text{Work done} = -4500 \times 0.025\,\text{J} = -112.5\text{ J}$$
(*note negative as work done by the force has to be defined in the direction of the force*)

$$\text{Initial KE} = \frac{1}{2}mu^2 = 0.5 \times 0.03 \times 200^2 = 600\text{ J}$$

A loss of 112.5 J will not reduce the KE to below 0 so the bullet will still be moving on exit.

As the work done is equal to the change in kinetic energy,

$$-112.5 = \frac{1}{2}mv^2 - 600$$

(example continues on the next page)

***Elastic potential energy (EPE) and Hooke's Law* (AQA ONLY)**

Another form of potential energy, which AQA students require in M2, is elastic potential energy. This is the energy stored in a spring when it is stretched or compressed. $EPE = \frac{\lambda x^2}{2l}$. An object attached to the spring has the potential to move when the spring is released. This is often used in conjunction with Hooke's Law. The law is used in the form $T = kx$ where k is the stiffness of the string/ spring, or in the form $T = \frac{\lambda x}{l}$, where l is the natural length and λ is the modulus of elasticity. Please refer to your notes and recommended text for example questions.

Energy

Kinetic energy and gravitational energy are both forms of mechanical energy. When gravity is the only force acting on a body, total mechanical energy is always conserved.

When solving problems involving a change in vertical position, it is often convenient to use the work-energy principle (the total work done by the forces acting on a body is equal to the increase itn he kinetic energy of the body) in a slightly different form.

That is:
Work done by external forces other than weight
= *mgh* + increase in KE
= increase in GPE + in crease in KE
= increase in total mechanical energy

Power

Power is the rate of doing work

The power of a vehicle moving at speed *v* under a driving force *F* is given by *Fv*. Power is measured in Watts (W).

For a motor vehicle it is the engine which produces the power, whereas for a cyclist riding a bike it is the cyclist.

Solving for *v*

$$\frac{1}{2}mv^2 = 600 - 112.5$$

$$v^2 = \frac{2(600-112.5)}{0.03}$$

$$v = 180 \text{ ms}^{-1}$$

So the bullet will exit with a speed of 180 ms^{-1}

Example *(GPE)*

Calculate the gravitational potential energy, relative to the ground, of a small ball of mass 0.2 kg at a height of 1.6 m above the ground.

Mass $m = 0.2$, height $h = 1.6$

$$\text{GPE} = mgh = 0.2 \times 9.8 \times 1.6 = 3.14 \text{ J}$$

If the ball falls then:
Loss in PE = work done by gravity = gain in KE
(There is no change in the total energy (KE + PE) of the ball.)

Note. The examples here could be classified as simple examples. It should be recognised that examination questions on Energy are likely to be much more involved, however, due to the space limitation here that level of example could not be posed. Please see your notes and recommended text for further examples

Example *(Power)*

A car of mass 1100 kg can produce a maximum power of 44000 W. The driver of the car wishes to overtake another vehicle. If air resistance is ignored, what is the maximum acceleration of the car when it is travelling at 30 ms^{-1} ?

Power = force x velocity
The driving force at 30 ms^{-1} is F N, where
44000 = F x 30

$$F = \frac{44000}{30} = \frac{4400}{3} \approx 1466.66 \text{ N}$$

By Newton's second law $F = ma$

$$\text{acceleration} = \frac{1466.66}{1100} = \frac{4}{3} \approx 1.33 \text{ ms}^{-2}$$

REVISION SHEET – MECHANICS 2

EQUILIBRIUM OF A RIGID BODY

The main ideas are

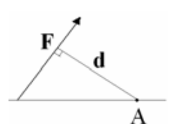
	AQA	Edx	MEI	OCR
Moments	M2	M2	M2	M2
Toppling/ Sliding	M4	M3	M2	M2
Frameworks	M4	-	M2	-

Before the exam you should know:

- What moments are and how to calculate them.
- How to calculate whether an object will slide or topple first.
- How to use moments and resolving in situations involving frameworks.

Moments (moment of a force)

The moment of the force, **F**, about an axis through A, perpendicular to the plane containing A and **F**, is Fd.



Where there is a hinge or a fulcrum there is always some kind of reaction force at the hinge or fulcrum. This is why it often makes sense to take moments about a hinge or a fulcrum, as the reaction has no moment about that point.

Remember to use the principle that, under the action of coplanar forces, a rigid body is in equilibrium if and only if: the vector sum of the forces is zero, and the sum of the moments of the forces about any point is zero.

Many problems can be solved by using a combination of resolving forces and taking moments.

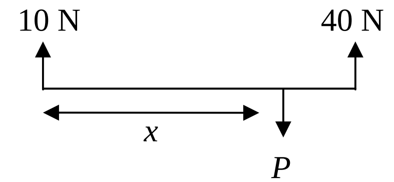
Always draw a diagram **–** if you try to work without a diagram, you are very likely to make mistakes with signs, or to miss out forces.

Remember to include reaction forces at a support or hinge in the force diagram. These have no effect when you take moments about the support or hinge, but you need to take them into account when you resolve forces or take moments about a different point.

Practice the more difficult examples which have many forces involved, e.g. a ladder on a rough surface placed against a rough wall etc.

Examples *(Moments)*

1. What is the distance, x, required for a light horizontal rod, of length 1.5 m to be resting in equilibrium?

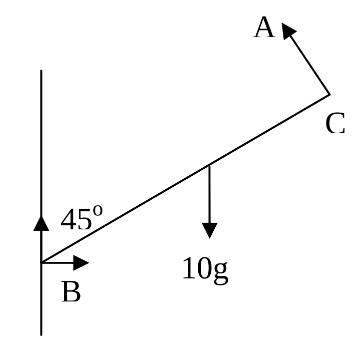


Resolving vertically: $10 + 40 - P = 0$, $\Rightarrow$ $P = 50$ N

Taking moments about left-hand end:

$$40 \times 1.5 - Px = 0$$
$$50x = 60$$
$$x = 6/5 = 1.2 \text{ m}$$

2. A uniform rod of mass 10g is pivoted at B and held at an angle of 45° to the vertical by a force applied at C, perpendicular to BC. What is the force A?



If the length of the rod is said to be $2x$, then taking moments about B (as both the forces at B go through that point)

$$10gx\cos 45^o = A \times 2x$$

$$A = 10g\cos 45^o / 2 = 34.65\,\text{N}$$

Toppling/Sliding

When an object rests on a surface the only forces which are acting are its weight and the resultant of all the contact forces between the surfaces. This resultant is normally resolved into two components, the Friction, which acts parallel to the possible motion (opposing the motion) and the normal reaction, which acts perpendicular to the Friction.

Consequently, there are two ways in which equilibrium could be broken,

i) the object is on the point of sliding
ii) the object is on the point of toppling

You should be familiar with the process of checking for each of these two possibilities.

See the example to the right for the detail of calculations to determine when that block will slide or topple. Note you will need to use trigonometry to find when an object topples.

You should be comfortable with the idea that on the point of sliding $F = \mu R$ and be able to use this to solve equations involving the three variables.

Frameworks *(MEI only in M2)*

The general strategy for answering these questions includes:

1. To consider external forces, for example reaction forces and weights. Reactions are normally best represented as components.
   a) Resolve horizontally and vertically
   b) Take moments about some suitable point, if necessary.

(*Note:* it may not be necessary to find the external forces in some cases.)

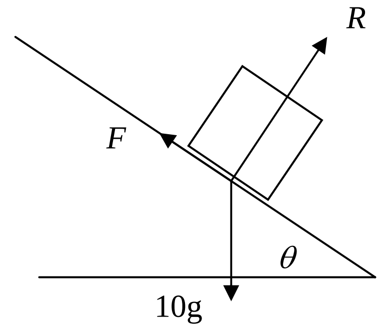
2. Put the internal forces in each rod: draw a force from each end of the rod.
(Note: you may be able to deduce from the diagram whether each rod is in tension or thrust, in which case you can draw the arrows in the correct direction. If you cannot be certain of all the directions, you may find it easier to draw in the internal forces as if every rod were in tension)

Example *(Toppling/sliding)*

A cubic block of mass 12kg and side 15 cm is at rest on a rough slope. The coefficient of friction between the block and the slope is 0.6. If the slope is gradually increased will the block slide or topple?

The block is on the point of toppling when its weight acts vertically through its corner. By considering the geometry of the block it can be seen that this will occur when the angle reaches 45°.

The block is on the point of sliding when the frictional force up the slope is at its maximum possible value of μR.



Resolving perpendicular to the slope:

$$R = 12g\cos\theta$$

Resolving parallel to the slope:

$$F = \mu R = 12g\sin\theta$$

$$\therefore \quad 0.6 \times 12g\cos\theta = 12g\sin\theta$$

$$\tan\theta = 0.6$$

$$\theta = 31^o$$

i.e. the block will slide at an angle of 31^o and will topple at an angle of 45^o, so the block will slide.

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3. Choose a point on the framework where there are only one or two unknown forces. Resolve horizontally and/or vertically to find these unknown forces. Go on to another point, and continue until you have found all the internal forces.

(Note: in most cases that you meet, the angles involved are likely to be simple ones such as 30°, 45° and 60°. When you calculate each force, you should write down the exact value using surds. If you need to use a result in another calculation, you can then use this exact value. Answers can then be given to a suitable degree of accuracy if required.)

REVISION SHEET – MECHANICS 2

MOMENTUM, RESTITUITION & IMPULSE

The main ideas are

	AQA	Edx	MEI	OCR
Momentum	M1	M1	M2	M1
Conservation law	M1	M1	M2	M1
Coefficient of Restitution	M3	M2	M2	M2
Impulse	M3	M2	M2	M2

Before the exam you should know:

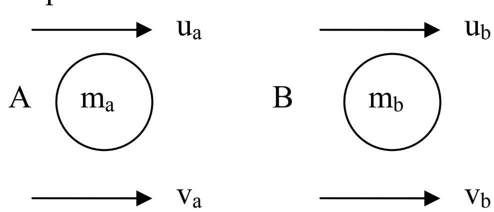
- What momentum is, in particular that it is a vector and how to calculate it.
- About conservation of momentum and understand the calculations involved.
- About the coefficient of restitution.
- What impulse is and the two ways in which it can be calculated.

Momentum

If an object of mass m has velocity $\mathbf{v}$, then the momentum of the object is its mass x velocity,
i.e. Momentum = $m\mathbf{v}$
Momentum is a vector quantity because it has both a magnitude and direction associated with it. Its units are kg ms^{-1} or Newton seconds Ns.

Conservation law

For a system of interacting particles, the total momentum of a system remains constant when there is no resultant external force acting.
If two particles collide:



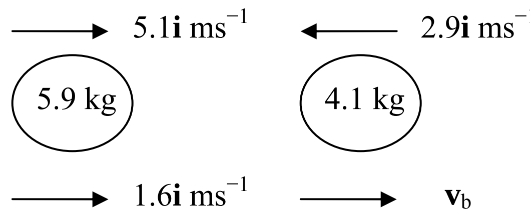
Where:
- m_a = mass of particle A
- m_b = mass of particle B
- $\mathbf{u}_a$ = velocity of particle A before collision
- $\mathbf{u}_b$ = velocity of particle B before collision
- $\mathbf{v}_a$ = velocity of particle A after collision
- $\mathbf{v}_b$ = velocity of particle B after collision

Then, the principle of conservation of momentum is: $m_a\mathbf{u}_a + m_b\mathbf{u}_b = m_a\mathbf{v}_a + m_b\mathbf{v}_b$
i.e. Total momentum before collision
= Total momentum after collision
It is essential to read these questions carefully noting the directions of the velocities of the particles both before and after a collision. Remember two particles could coalesce and move as one particle.

Example *(Momentum)*
A biker and their motorbike have a combined mass of 320 kg and are travelling along a straight horizontal road at 22 ms^{-1}. A cyclist and their cycle, which have a combined mass of 80 kg, travel in the opposite direction at 10 ms^{-1}. What is the momentum of the biker and what is the momentum of the cyclist?

Defining the direction of the biker as positive, then:
momentum of biker = $320 \times 22 = 7040$ kg ms^{-1}
momentum of cyclist = $80 \times (-10) = -800$ kg ms^{-1}

Example *(Conservation law)*
A particle, of mass 5.9 kg, travelling in a straight line at $5.1\mathbf{i}$ ms^{-1} collides with another particle, of mass 4.1 kg, travelling in the same straight line, but in the opposite direction, with a velocity of $2.9\mathbf{i}$ ms^{-1}. Given that after the collision the first particle continues to move in the same direction at $1.5\mathbf{i}$ ms^{-1}, what velocity does the second particle move with after the collision?



$$m_a\mathbf{u}_a + m_b\mathbf{u}_b = m_a\mathbf{v}_a + m_b\mathbf{v}_b$$
$$5.9 \times 5.1\mathbf{i} + 4.1 \times (-2.9\mathbf{i}) = 2.9 \times 1.6\mathbf{i} + 4.1 \times \mathbf{v}_b$$
$$30.09\mathbf{i} - 11.89\mathbf{i} - 4.64\mathbf{i} = 4.1 \times \mathbf{v}_b$$
$$13.56\mathbf{i} = 4.1\,\mathbf{v}_b$$
$$\mathbf{v}_b = 3.31\mathbf{i}\ \text{ms}^{-1}$$

Coefficient of Restitution (CoR)

The coefficient of restitution is conventionally denoted by e and is a constant between 0 and 1. It relates the speed before and after a direct collision between two bodies and is taken to be the ratio:

$$\frac{\text{speed of separation}}{\text{speed of approach}} = e$$

It is also known as Newton's law of impact and can be written:
Speed of separation = e x speed of approach

A collision for which $e = 1$ is called perfectly elastic, and a collision for which $e = 0$ is called perfectly inelastic.

In answering a question on collisions you may be asked to calculate a number of things that are all related, in particular, the speed of a particle before/ after a collision, the coefficient of restitution for the collision and finally the loss in mechanical energy due to the collision.

For the MEI specification you may also be asked about oblique collisions, e.g. a ball bouncing off a snooker table's cushion.

Impulse

The impulse of a force **F**, acting for a short time, t, on a body is the quantity **F**t,

i.e. Impulse = **F**t

In general the impulse of **F** is given by $\int \mathbf{F}\,dt$.

For motion in one dimension it can be shown that:

The impulse of **F** = Change in momentum,
= $m\mathbf{v} - m\mathbf{u}$

Where **u** is the initial velocity and **v** is the final velocity.

Impulse is often found indirectly by reviewing the change in momentum. This is very useful when the force and time are unknown but both the momentum before and after are know (or can be easily found).

Example *(CoR)*

For the example given for the conservation law (see previous page), what is the value of the coefficient of restitution and what is the loss in kinetic energy?

Speed of approach = 5.1 – (–2.9) = 8

Speed of separation = 3.31 – 1.6 = 1.71

$$e = \frac{1.71}{8} = 0.21$$

Note that this is equivalent to $e = \dfrac{v_b - v_a}{u_a - u_b}$ from the notation introduced for the conservation law.

$$\text{K.E. before impact} = \frac{1}{2}(5.9)(5.1)^2 + \frac{1}{2}(4.1)(-2.9)^2$$
$$= 93.97$$

$$\text{K.E. after impact} = \frac{1}{2}(5.9)(1.6)^2 + \frac{1}{2}(4.1)(3.31)^2$$
$$= 30.01$$

K.E. lost = 93.97 – 30.01
= 63.96

Example *(Impulse)*

A goalkeeper kicks a stationary football, of mass 0.6 kg, which then moves with a velocity of 35 ms^{-1}. What impulse does the goalkeeper impart on the football?

Using the formulae:

Impulse = Change in momentum
= final momentum − initial momentum
= $0.6 \times 35 - 0.6 \times 0$
= 21 kg ms^{-1}

It is interesting to note that this momentum could be produced in a number of ways,

e.g. by a force of 100 N acting for 0.21 seconds *or*
by a force of 25 N acting for 0.84 seconds.

REVISION SHEET – MECHANICS 2

PROJECTILES

The main ideas are

	AQA	Edx	MEI	OCR
The maximum height of a projectile	M1	M2	M1	M2
The range of a projectile	M1	M2	M1	M2
The path of a projectile	M1	M2	M1	M2

Before the exam you should know:

- You **must** be completely familiar and fluent with all of the constant acceleration equations, especially:

$$\mathbf{v} = \mathbf{u} + \mathbf{a}t$$

$$\mathbf{r} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$$

- You **must** be fluent with the use of vectors and resolving into horizontal and vertical components.
- The only force which acts on a projectile is gravity and we assume:
  - a projectile is a particle
  - it is not powered
  - the air has no effect on its motion
- A projectile experiences a constant acceleration of $g = 9.8\text{ms}^{-2}$ vertically downwards
- The horizontal component of acceleration is 0 for a projectile, so its horizontal component of velocity is **constant**.
- If a projectile has an initial speed u, at an angle of θ to the horizontal, its initial velocity is $\mathbf{u} = \begin{pmatrix} u\cos\theta \\ u\sin\theta \end{pmatrix}$ and its acceleration is $\begin{pmatrix} 0 \\ -g \end{pmatrix}$
- At maximum height, the vertical component of a projectile's velocity, v_y, is 0.
- Know how to derive the equation of the path of a projectile: $y = x\tan\theta - \frac{gx^2}{2u^2}\left(1+\tan^2\theta\right)$

Finding the maximum height of a projectile

Example

A ball is kicked with a speed of $15\,\text{ms}^{-1}$ over level ground at an angle of 40° to the horizontal. What is the maximum height reached?

Solution

We are concerned with the vertical component of the ball's motion.

$u_y = 15\sin 40^\circ$

$a_y = g = -9.8$

$v_y = 0$ (at maximum height)

$y = ?$ (this is the maximum height we wish to find)

Choose the appropriate equation of motion, based on the information you have and what you need to calculate:

$v^2 = u^2 + 2as$

$$y = \frac{v_y^2 - u_y^2}{2a_y}$$

$$= \frac{0-\left(15\sin 40^\circ\right)^2}{2\times -9.8}$$

$= 4.74\text{m}$ (3 s.f.)

so the maximum height of the ball is 4.74m (3 s.f.)

Finding the range of a projectile

Continuing the example used for maximum height, if the ball is kicked over level ground, the point where it lands will have vertical displacement, y, of 0. Its range is its horizontal displacement, x, from its starting point at the point where it lands. To calculate the ball's range, you can calculate the time it takes to return to the ground and then use this time to calculate the horizontal displacement at that time.

Question: What is the range of the ball?

Solution

Considering the vertical motion:

$u_y = 15\sin 40^\circ$

$a_y = -9.8$

$y = 0$ (when the ball returns to the ground)

$t = ?$ (when the ball returns to the ground)

Choose the appropriate equation of motion, based on the information you have and what you need to calculate:

$$s = ut + \frac{1}{2}at^2$$

$$y = t\left(u_y + \frac{1}{2}a_y t\right) \Rightarrow 0 = t\left(15\sin 40^\circ - 4.9t\right)$$

$$\Rightarrow t = 0 \text{ or } t = \frac{15\sin 40^\circ}{4.9} = 1.968 \text{ seconds (4 s.f.)}$$

$t = 0$ is when the ball left the ground, so it lands when $t = 1.968$ seconds (4s.f.)

Now considering the horizontal motion:

$u_x = 15\cos 40^\circ$

$a_x = 0$

$x = ?$ (the range)

$t = 1.968$

(when the ball returns to the ground, calculated above)

Choose the appropriate equation of motion, based on the information you have and what you need to calculate:

$$s = ut + \frac{1}{2}at^2$$

$$x = t\left(u_x + \frac{1}{2}a_x t\right) \Rightarrow x = 1.968\left(15\cos 40^\circ + 0\right) = 22.6\text{m (3s.f.)}$$

The range of the ball is 22.6m (3 s.f.)

Finding the path of a projectile

Sticking with the same example:

Question: Derive the equation of the path of the ball, assuming it starts at the origin.

Solution

Using $s = ut + \frac{1}{2}at^2$ with $u_x = 15\cos 40^\circ$ and $u_y = 15\sin 40^\circ$ gives:

$$x = \left(15\cos 40^\circ\right)t \quad [1] \text{ and } y = \left(15\sin 40^\circ\right)t - 4.9t^2 \quad [2]$$

[1] $\Rightarrow t = \dfrac{x}{15\cos 40^\circ}$. Substituting for t in [2] gives

$$y = x\tan 40^\circ - 4.9 \times \frac{x^2}{\left(15\cos 40^\circ\right)^2}$$

The path of the ball is the parabola $y = 0.839x - 0.0371x^2$ (3 s.f.)

REVISION SHEET – MECHANICS 2

VARIABLE ACCELERATION USING DIFFERENTIATION AND INTEGRATION

The main ideas are

	AQA	Edx	MEI	OCR
Differentiation	M2	M2	M1	M1
Integration	M2	M2	M1	M1
Differentiation in 2 dimensions	M2	M2	M1	M1
Integration in 2 dimensions	M2	M2	M1	M1

The main ideas in this topic are:

- **Using differentiation and integration to obtain expressions for the displacement, velocity and acceleration from one another. You should be able to do this in one, two or three dimensions.**
- **Obtaining values of associated quantities such as speed and distance travelled.**

Before the exam you should know:

- **Velocity is the rate of change of displacement.** Therefore to obtain an expression for a particle's velocity at time *t* you should differentiate the expression for its displacement.
- **Acceleration is the rate of change of velocity.** Therefore to obtain an expression for a particle's acceleration at time *t* you should differentiate the expression for its velocity.
- Reversing the two ideas above, **a particle's velocity can be obtained by integrating the expression for its acceleration** and **a particle's displacement can be obtained by integrating the expression for its velocity**. In both cases this will introduce a constant of integration whose value can be found if the particle's displacement or velocity is known at some particular time.
- The above facts apply to both:
  1. particles travelling in one dimension. In this case each of the displacement, velocity and acceleration is a (scalar valued) function of time, all of which can be differentiated and integrated in the usual way.
  2. particles travelling in two and three dimensions, when the displacement, velocity and acceleration are all vectors with components dependent on *t* (time). We differentiate and integrate such expressions in the usual way, dealing with each component separately. There are several examples of this on this sheet.
- You should be comfortable with both column vector and **i**, **j**, **k** notation for vectors.
- (*For AQA and Edexcel*) How to solve a differential equation by separating the variables and then solve using integration techniques. (please refer to your notes on this topic)

Using differentiation - a particle travelling in a straight line.

***Example**: An object moves in a straight line, so that its displacement relative to some fixed origin at time t is given by* $s = t^3 - 5t^2 + 4$.

1. *Find expressions for its velocity and acceleration at time t.*
2. *Calculate the velocity and acceleration of the object when t = 0 and when t = 1.*
3. *What is the displacement of the object when its velocity is zero?*

Solution.

1. $s = t^3 - 5t^2 + 4$ so that $v = \dfrac{ds}{dt} = 3t^2 - 10t$ and $a = \dfrac{dv}{dt} = 6t - 10$.
2. When $t = 0, v = 3\times 0 - 10\times 0 = 0\text{ms}^{-1}$ and $a = 6\times 0 - 10 = -10\text{ms}^{-2}$ and when $t = 1, v = 3\times 1 - 10\times 1 = -7\text{ms}^{-1}$ and $a = 6\times 1 - 10 = -4\text{ms}^{-2}$.
3. The velocity of the object at time t is $3t^2 - 10t = t(3t - 10)$. This is zero when t = 0 or when $t = \dfrac{10}{3}$. The displacement of the particle when t = 0 is $s = 4\text{m}$ and then displacement of the particle when $t = \dfrac{10}{3}$ is

Using integration – a particle travelling in a straight line.

Example*: An object is moving in a straight line with acceleration at time t given by* $a = 10 - 6t$.
Given that when t = 1, s = 0 and v = –5, where s is the object's displacement and v is the object's velocity, find an expression for v and s in terms of t.
Hence find out the displacement of the particle when it first comes to rest.

Solution

$v = \int a \ \mathrm{d}t = \int (10 - 6t)\mathrm{d}t = 10t - 3t^2 + c$. But when $t = 1$, $v = 5 = 10 - 3 + c \Rightarrow c = -2$. So $v = 10t - 3t^2 - 2$.

$s = \int v \ \mathrm{d}t = \int (10t - 3t^2 - 2)\ \mathrm{d}t = 5t^2 - t^3 - 2t + c$. But when $t = 1$, $s = 0 = 5 - 1 - 2 + c \Rightarrow c = -2$. So $s = 5t^2 - t^3 - 2t - 2$

Using differentiation – an example in two dimensions using column vector notation.

Example*: A girl throws a ball and, t seconds after she releases it, its position in metres relative to the point where she is standing is modelled by*

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15t \\ 2 + 16t - 5t^2 \end{pmatrix}$$

where the directions are horizontal and vertical.

1. *Find expressions for the velocity and acceleration of the ball at time t.*
2. *The vertical component of the velocity is zero when the ball is at its highest point. Find the time taken for the ball to reach this point.*
3. *What is the speed of the ball when it hits the ground.*

Solution

1. The velocity is obtained by differentiating (with respect to t) the components in the vector giving the ball's position. This gives $v = \begin{pmatrix} 15 \\ 16 - 10t \end{pmatrix}$. The acceleration is obtained by differentiating (with respect to t) the components in the vector giving the ball's velocity. This gives $a = \begin{pmatrix} 0 \\ -10 \end{pmatrix}$.
2. The vertical component of the velocity is 16 – 10t. This is zero when $t = \frac{16}{10} = \frac{8}{5} = 1.6$ seconds.
3. The ball hits the ground when the vertical component of the balls position is zero. In other words when $2 + 16t - 5t^2 = 0$. Rearranging this as $5t^2 - 16t - 2 = 0$ and then using the formula for the solutions of a quadratic we see that the solutions of this are $t = -0.12$ and $t = 3.3$ (to 2 s.f). Clearly the value we require is t = 3.3. The velocity of the ball when t = 3.3 is $\begin{pmatrix} 15 \\ -17 \end{pmatrix}$ and so the speed is $\sqrt{15^2 + (-17)^2} = 22.67 \text{ ms}^{-1}$.

Using Integration and Newtons 2$^{\text{nd}}$ Law an example in 2 dimensions with i, j notation.

Example*: A particle of mass 0.5 kg is acted on by a force, in Newtons, of* $\mathbf{F} = t^2\mathbf{i} + 2t\mathbf{j}$. *The particle is initially at rest and t is measured in seconds.*

1. *Find the acceleration of the particle at time t.*
2. *Find the velocity of the particle at time t.*

Solution

Newton's second law, $\mathbf{F} = m\mathbf{a}$ gives that $\mathbf{F} = t^2\mathbf{i} + 2t\mathbf{j} = 0.5\mathbf{a}$ so that $\mathbf{a} = 2t^2\mathbf{i} + 4t\mathbf{j}$.

We have that $\mathbf{v} = \int \mathbf{a}\ \mathrm{d}t = \left(\frac{2t^3}{3} + c\right)\mathbf{i} + \left(2t^2 + d\right)\mathbf{j}$ where c and d are the so-called "constants of integration".

We are told that the particle is at rest when t = 0 and so $c = d = 0$. This gives $\mathbf{v} = \frac{2t^3}{3}\mathbf{i} + 2t^2\mathbf{j}$.